2 Holomorphic Functions

In this chapter we discuss functions of a complex variable and what it means for such to be *differentiable.* This turns out to be much more subtle and restrictive than in real analysis.

2.1 Functions of a Complex Variable

A function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ may be defined in the obvious manner, using a *rule.*

Example 2.1. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^3 - z$. This evaluates as expected; e.g.

$$f(2+i) = (2+i)^3 - (2+i) = 2^3 + 3 \cdot 2^2 i + 3 \cdot 2i^2 + i^3 - 2 - i = 10i$$

As in real analysis, it is common to express a function simply using a rule: its *implied domain* is the largest possible set $D \subseteq \mathbb{C}$ on which the rule is defined.

Example 2.2. $f(z) = \frac{1}{z^2+9}$ has implied domain $D = \mathbb{C} \setminus \{\pm 3i\}$.
The function may also be expressed in two other common ways.
Real and imaginary parts: Write $f(z) = u(x,y) + iv(x,y)$ where $u, v : D \to \mathbb{R}$. In this case,

$$f(z) = \frac{1}{(x+iy)^2+9} = \frac{1}{x^2 - y^2 + 9 + 2ixy} = \frac{x^2 - y^2 + 9}{(x^2 - y^2 + 9)^2 + 4x^2y^2} + i \frac{-2xy}{(x^2 - y^2 + 9)^2 + 4x^2y^2}$$

Polar form: Write $z = re^{i\theta}$ to obtain

$$f(z) = \frac{1}{r^2 e^{2i\theta} + 9}$$

Depending on the function, each of these approaches might have certain advantages or disadvantages. You might also wish to combine the approaches: for instance writing u, v as functions of r, θ.

Examples 2.3. We consider the function $f(z) = z^2$ and some relations in more detail.

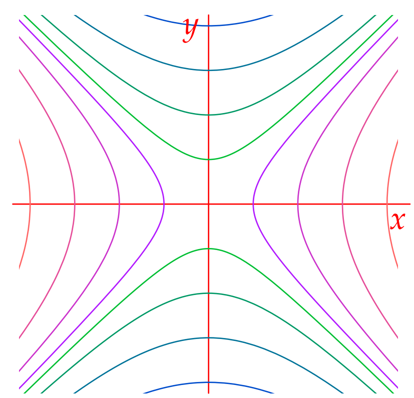
First we write f in the three standard forms of Example 2.2,

$$f(z) = z^2 = x^2 - y^2 + 2ixy = r^2 e^{2i\theta}$$

We cannot straightforwardly *graph* $f(z) = z^2$ (don't think about a parabola!). However, we can visualize the real and imaginary parts

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy$$

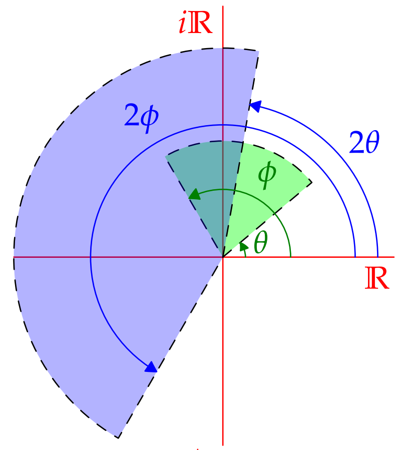
as graphs of functions $\mathbb{R}^2 \to \mathbb{R}$. Both are *saddle surfaces* and can be analyzed using the standard tools of multivariable calculus. For instance, the *level curves* $u =$ constant and $v =$ constant are *hyperbolæ.*



Level curves of $u = x^2 - y^2$

Now consider the polar form and what it means for the argument:

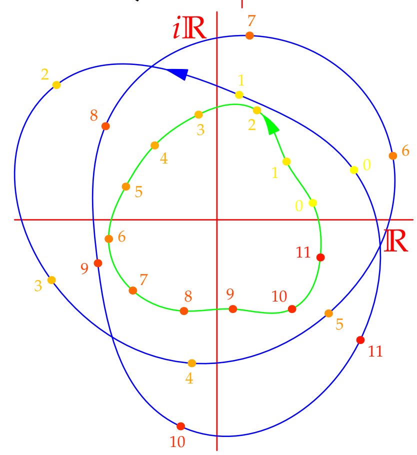
$$f(z) = r^2 e^{2i\theta} \implies \arg(z^2) = 2\arg z$$



This can be visualized by considering sectors of the plane: the sector between arguments θ and ϕ is *doubled* in angle to the sector between 2θ and 2ϕ.

Away from the origin, $f(\pm z) = z^2$ shows that f is a *two-to-one* function: f maps the sector with $\arg z \in [0, \pi)$ onto the entire plane.

We can also visualize this pathwise. If z traces a path around the origin, z^2 traces a new path *twice* round the origin! The colored dots on the two paths correspond under $z \mapsto z^2$.



Similar behavior occurs with higher powers. For instance, $z \mapsto z^3$ maps a single loop round the origin to a *triple* loop; away from the origin we have a *three-to-one* function. The function $z \mapsto z^n$ is an *n-to-one* map.

By contrast, the principal square root function $\sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ *halves* the principal argument $\theta = \operatorname{Arg} z$, from which $\arg\sqrt{z} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Written instead using fractional exponent notation, we see that

$$z \mapsto z^{1/2} = \{\sqrt{r}e^{\frac{i\theta}{2}}, -\sqrt{r}e^{\frac{i\theta}{2}}\}$$

is a *two-valued* function.

Basic 2D geometry with complex numbers

Complex functions can be used to describe the primary geometric transformations of the plane.

Translation If w is constant, the function $f(z) = z + w$ translates the plane, shifting the origin to w.

Scaling If $R \in \mathbb{R}$ is constant, the function $f(z) = Rz$ scales the complex plane, inflating the distance from the origin.

Rotation Given ϕ, the function $f(z) = e^{i\phi}z$ rotates the complex plane by ϕ radians counter-clockwise around the origin. The easy way to see this is to write the function in polar form itself:

$$f(z) = f(re^{i\theta}) = e^{i\phi}re^{i\theta} = re^{i(\theta+\phi)}$$

Reflection Complex conjugation $f(z) = \overline{z} = x - iy$ reflects the complex plane in the horizontal axis. Combining with rotation, we can produce the reflection in any line through the origin. To reflect in the line through 0 and w with $\operatorname{Arg} w = \phi$;

1. Rotate the plane by $-\phi$: $z \mapsto e^{-i\phi}z$
2. Reflect in the real axis: $z \mapsto \overline{e^{-i\phi}z} = e^{i\phi}\overline{z}$
3. Rotate the plane back by ϕ: the result is the function $f(z) = e^{2i\phi}\overline{z}$

Combining these functions allows one to rotate around any point and reflect across any line.

Examples 2.4. 1. The function $f(z) = e^{\frac{i\pi}{3}}(z-2i)+2i$ rotates the complex plane by $\frac{\pi}{3}$ radians around the point $2i$.

2. We compute the function that reflects across the line joining $\alpha = 2+i$ and $\beta = 4+3i$.

   Since $\beta - \alpha = 2+2i$ has argument $\phi = \frac{\pi}{4}$, we combine reflection across the line making ϕ through the origin with translation by α (translate by $-\alpha$, reflect, translate back by α):

$$\begin{aligned} f(z) &= e^{2i\phi}(\overline{z-\alpha}) + \alpha = e^{\frac{i\pi}{2}}(\overline{z-2-i}) + 2 + i = i(\overline{z} - 2 + i) + 2 + i \\ &= i\overline{z} + 1 - i \end{aligned}$$

   As a sanity check, you should verify that $f(\alpha) = \alpha$ and $f(\beta) = \beta$: why?

Exercises 2.1 1. For each function, describe its implied domain (page 1).

   (a) $f(z) = \dfrac{1}{4+z^2}$ (b) $f(z) = \dfrac{z-1}{e^z-1}$ (c) $f(z) = \dfrac{z^2+z+1}{z^4-1}$

2. Write the function in terms of its real and imaginary parts: $f(z) = u(x,y) + iv(x,y)$.

   (a) $f(z) = z^3 - 4z^2 + 2$ (b) $f(z) = \dfrac{z^2}{1-\overline{z}}$ (c) $f(z) = e^{\overline{z}}$

3. Write the function $f(z) = \dfrac{1}{|z|^2}\overline{z}$ in polar form.

4. Find an expression for the function which reflects across the vertical line through the point $\alpha = -1$.

5. For Example 2.42 evaluate the function $g(z) = e^{2i\phi}(\overline{z-\beta}) + \beta$. Why are you not surprised by the result?

6. Let $\phi = \tan^{-1}\frac{3}{4}$. Find the result (in rectangular co-ordinates) of rotating $z = -2+i$ counter-clockwise by ϕ radians around the origin.
   (*Hint: consider a 3:4:5 triangle!*)

7. Prove, using the expressions on page 2, that the composition of two reflections is a rotation, and that the composition of a rotation and a reflection is a reflection.

2.2 Open sets, Limits and Continuity

Before we can differentiate, we need to understand limits and continuity. Much of this section should be treated as a reference, where we review without proof several preliminaries which are identical to (multivariable) real analysis.

Definition 2.5 (Sequences and Limits). Let $(z_n) = (z_1, z_2, \ldots)$ be a *sequence* of complex numbers.

1. (z_n) *converges* to z_0, and we write $\lim_{n\to\infty} z_n = z_0$ or simply $z_n \to z_0$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z_0| < \epsilon$$

2. (z_n) is *Cauchy* if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |z_m - z_n| < \epsilon$$

Theorem 2.6. 1. *Cauchy completeness: (z_n) converges if and only if it is Cauchy.*

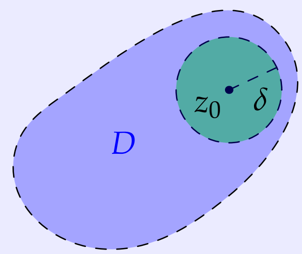
2. *Bolzano–Weierstraß: If (z_n) is bounded, then it has a convergent subsequence.*

Definition 2.7 (Disks, Sets and Neighborhoods). The *open disk* (or *δ-ball*) centered at $z_0 \in \mathbb{C}$ with radius δ is the set

$$B_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

The *punctured open disk* centered at z_0 with radius δ is the set

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$$



An open neighborhood of z_0

Let $D \subseteq \mathbb{C}$ be a subset.

- D is *open* if every point is *interior*: at every point we can center an open disk $B_\delta(z_0) \subseteq D$:

$$\forall z_0 \in D,\ \exists \delta > 0 \text{ such that } |z - z_0| < \delta \implies z \in D$$

- D is a *neighborhood* of z_0 if it contains some $B_\delta(z_0)$. A neighborhood can, but need not be, open. A *punctured neighborhood* instead contains a punctured disk.
- D is *closed* if its complement $\mathbb{C} \setminus K$ is open.

Theorem 2.8. *K is closed if and only if every Cauchy sequence $(z_n) \subseteq K$ has its limit in K.*

We define continuity of functions in the usual way.

Definition 2.9 (Continuity). Let $f : D \to \mathbb{C}$ and $z_0 \in D$. We say that f is *continuous at z_0* if

$$\forall \text{ sequences } (z_n) \subseteq D \text{ with } z_n \to z_0 \text{ we have } f(z_n) \to f(z_0)$$

f is *continuous* (on D) if it is continuous at all points $z_0 \in D$.

As in real analysis, it is helpful to relate this to limits of functions.

Definition 2.10 (Limits of Functions). Let $f : D \to \mathbb{C}$, where D contains an open punctured neighborhood of z_0. We say that w_0 is the *limit of f as z approaches z_0* and write $\lim_{z\to z_0} f(z) = w_0$, if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

Theorem 2.11. *If z_0 is an* interior *point to D, then $f : D \to \mathbb{C}$ is continuous at z_0 if and only if*

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Otherwise said, for any ϵ-ball $B_\epsilon\big(f(z_0)\big)$, there exists a δ-ball $B_\delta(z_0)$ such that $f\big(B_\delta(z_0)\big) \subseteq B_\epsilon\big(f(z_0)\big)$. We won't spend much time on calculations since these tend to proceed similarly to real analysis.

Example 2.12. We show that $f(z) = z^2$ is continuous by proving that $\lim_{z\to z_0} z^2 = z_0^2$ for all $z_0 \in \mathbb{C}$.

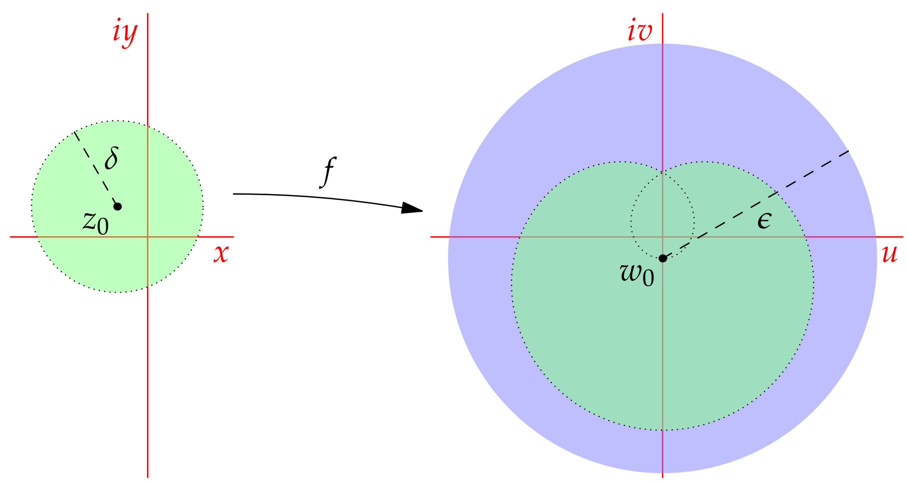
Let $z_0 \in \mathbb{C}$ and $\epsilon > 0$ be given. Define $\delta = \min\{1, \frac{\epsilon}{1+2|z_0|}\}$. By the triangle-inequality,

$$|z - z_0| < \delta \implies |z + z_0| = |z - z_0 + 2z_0| \le |z - z_0| + 2|z_0| < \delta + 2|z_0| \le 1 + 2|z_0|$$

from which

$$|z - z_0| < \delta \implies \left|z^2 - z_0^2\right| = |z - z_0|\,|z + z_0| < \delta(1 + 2|z_0|) \le \epsilon$$

The picture below should help you visualize this. Given the ϵ-ball centered at $w_0 = z_0^2$, we've described how to choose $\delta > 0$ so that the punctured δ-ball centered at z_0 is mapped to a region *inside* the original ϵ-ball.



The picture illustrates the case when

$$z_0 = \frac{1}{2}e^{\frac{3\pi i}{4}} = \frac{1}{2\sqrt{2}}(-1+i), \quad w_0 = -\frac{i}{4}, \quad \epsilon = \frac{5}{2} \quad \text{and} \quad \delta = \min\left\{1, \tfrac{5/2}{1+2\cdot\frac{1}{2}}\right\} = 1$$

Theorem 2.13 (Basic Limit Results). *Throughout, let $f, g : D \to \mathbb{C}$ be functions and $z_0 = x_0 + iy_0$ be a point satisfying the assumptions of Definition 2.10.*

1. *Limits are unique: If w_0 and $\widetilde{w_0}$ satisfy Definition 2.10, then $\widetilde{w_0} = w_0$.*
2. *If $f(z) = u(x,y) + iv(x,y)$ and $w_0 = u_0 + iv_0$, then*

$$\lim_{z\to z_0} f(z) = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

   In particular $\lim_{z\to z_0} \overline{z} = \overline{z_0}$

3. *Suppose $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$:*
   (a) *For any $a, b \in \mathbb{C}$, $\lim_{z\to z_0} (af(z) + bg(z)) = aw_0 + bw_1$*
   (b) *$\lim_{z\to z_0} (f(z)g(z)) = w_0 w_1$*
   (c) *$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$, provided $w_1 \neq 0$ and $g(z) \neq 0$ on a punctured neighbourhood of z_0.*
   (d) *If h is a function such that $\lim_{w\to w_0} h(w) = w_2$, then $\lim_{z\to z_0} h(f(z)) = w_2$*

Most of the above should feel familiar, and all should be intuitive. Parts 2 & 3 have obvious corollaries for continuous functions. For instance:

> f is continuous if and only if its real and imaginary parts $u, v : D \to \mathbb{R}$ are also.

Several of the properties in part 3 follow from part 2 by considering real and imaginary parts and the limit laws for functions $\mathbb{R}^2 \to \mathbb{R}$. For instance write $f(z) = u_1 + iv_1$ and $g(z) = u_2 + iv_2$, then

$$\lim_{z\to z_0} (f(z)g(z)) = \lim_{z\to z_0} (u_1u_2 - v_1v_2 + i(u_1v_2 + v_1u_2)) = w_0w_1$$

Examples 2.14. 1. $\lim_{z\to 1+3i} z^2 - i\overline{z} = (1+3i)^2 - i\overline{1+3i} = 1 + 6i - 9 - i(1-3i) = -11 + 5i$.

2. Every polynomial function is continuous on $\mathbb{C}$.
3. Every rational function $f(z) = \frac{p(z)}{q(z)}$ where p, q are polynomials, is continuous on its implied domain $\{z : q(z) \neq 0\}$.
4. Since the exponential, cosine and sine are continuous on $\mathbb{R}$, we see that the exponential function

$$\exp(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

   is continuous on $\mathbb{C}$.

Limits and the Point at Infinity

The treatment of infinity in complex analysis is very different to that in (single-variable) real analysis, where there are *two* infinities ($\pm\infty$), one for each *direction*. The convention in complex analysis is to have only one: that is, the sequences $z_n = n$ and $w_n = in$ diverge to the *same* infinity.

Definition 2.15. The *extended complex plane* or *Riemann sphere* is the set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where the symbol ∞ denotes the *point at infinity.*

A *neighborhood* of ∞ is any set containing an *open disk* at ∞; a subset of the form

$$\{\infty\} \cup \{z \in \mathbb{C} : |z| > M\}$$

We extend the notion of limit to the point at infinity.

1. $\lim_{z \to z_0} f(z) = \infty$ means: $\forall M > 0, \exists \delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z)| > M$
2. $\lim_{z \to \infty} f(z) = w_0$ means: $\forall \epsilon > 0, \exists N > 0$ such that $|z| > N \implies |f(z) - w_0| < \epsilon$
3. $\lim_{z \to \infty} f(z) = \infty$ means: $\forall M > 0, \exists N > 0$ such that $|z| > N \implies |f(z)| > M$

Example 2.16. We verify that $\lim_{z \to -3i} \frac{z^2}{z+3i} = \infty$.

Let $M > 0$ be given and define $\delta = \min\{1, \frac{4}{M}\}$. Then

$$\begin{aligned} 0 < |z + 3i| < \delta &\implies |z| \geq |3i| - |z + 3i| > 3 - \delta \geq 2 \\ &\implies \left| \frac{z^2}{z + 3i} \right| > \frac{4}{\delta} \geq M \end{aligned}$$

The intuitive relationships between functions, limits, zero and infinity feel like the dubious claims $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$!

Theorem 2.17. *Provided all limits make sense, Theorem 2.13 also applies to limits involving infinity. Moreover, we have th additional relationships:*

1. $\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
3. $\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$

Sketch Proof. Suppose $\lim_{z \to \infty} f(z) = w_0$. Simply let $\delta = \frac{1}{N}$ to see that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < \tfrac{1}{z} < \delta \implies \left| f(\tfrac{1}{z}) - w_0 \right| < \epsilon$$

The other five results are similar. ∎

Example 2.18. Consider $f(z) = \frac{5iz+1}{3z-2i}$. Plainly

$$\lim_{z\to\frac{2}{3}i}\frac{1}{f(z)} = \lim_{z\to\frac{2}{3}i}\frac{3z-2i}{5iz+1} = 0 \implies \lim_{z\to\frac{2}{3}i} f(z) = \infty$$

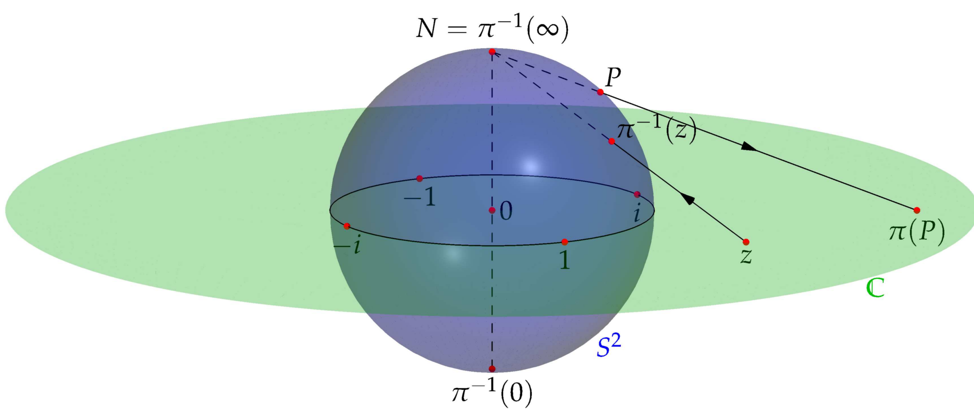
$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty}\frac{5i+1/z}{3-2i/z} = \frac{5i+\lim\frac{1}{z}}{3-2i\lim\frac{1}{z}} = \frac{5}{3}i$$

In view of these calculations, the function is really a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere to itself:

$$f(z) := \begin{cases} \frac{5iz+1}{3z-2i} & \text{if } z \neq \frac{2}{3}i, \infty \\ \infty & \text{if } z = \frac{2}{3}i \\ \frac{5}{3}i & \text{if } z = \infty \end{cases}$$

In fact f is a continuous bijection (exercise)!

For us, the Riemann sphere is merely a fun diversion. Unless indicated otherwise, all sets should be assumed to be subsets of the (*finite*) complex plane.[1] The Riemann sphere gets its name because $\overline{\mathbb{C}}$ can be visualized as a sphere S^2 where ∞ plays the role of the north pole. The rest of the sphere is identified bijectively with the equatorial plane $\mathbb{C}$ via *stereographic projection* $\pi : S^2 \to \overline{\mathbb{C}}$: in the picture, the image of $P \in S^2$ is the intersection $\pi(P)$ of $\mathbb{C}$ with the line through P and $N = \pi^{-1}(\infty)$.

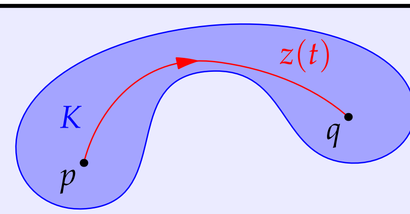


Compactness, Path-Connectedness & Continuity

We finish with two further properties we might wish our domains to have.

Definition 2.19. We say that a subset $K \subseteq \mathbb{C}$ is:

- *Compact* if it is closed and bounded.
- *Path-connected* if any two points can be joined by a path lying entirely within K: more precisely,



$$\forall p, q \in K,\ \exists \text{ continuous } z : [0,1] \to K \text{ such that } z(0) = p \text{ and } z(1) = q$$

[1] Computations of limits involving infinity are examinable, but the Riemann sphere and its interpretation are not.

What you should be familiar with are the analogues of these ideas in real analysis. Compactness generalizes the idea of a closed bounded interval: a compact subset of $\mathbb{R}$ is a union of finitely many closed bounded intervals. Path-connectedness generalizes the notion of an interval; that a set consists of only one piece.[2] In view of this, we translate over two familiar results from real analysis:

Extreme Value Theorem A continuous function on a closed bounded interval is bounded and attains its bounds.

Interval preservation A continuous function maps intervals to intervals. Essentially this is the intermediate value theorem.

Theorem 2.20. *Suppose $f : K \to \mathbb{C}$ is continuous.*

1. *If K is compact, so is the image $f(K)$.*
2. *If K is path-connected, so is $f(K)$.*

Proof.

1. The trick is to consider the *real-valued* function $|f|$.
   - Let $M = \sup\{|f(z)| : z \in K\}$. Since $|f|$ is real-valued, $\exists (z_n) \subseteq K$ such that $|f(z_n)| \to M$.
   - K is bounded; the Bolzano–Weierstraß theorem says (z_n) has a convergent subsequence (z_{n_k}) with limit z_0.
   - K is closed; thus $z_0 \in K$ and $f(z_0)$ is defined.
   - f is continuous; thus $f(z_{n_k}) \to f(z_0)$, necessarily $M = f(z_0)$ is finite and so $f(K)$ is bounded.
   - The closure of $f(K)$ is a short exercise.

   The same argument works even if K is a subset of the real numbers.
2. This is also an exercise.

■

We'll also have reason to use one final fact about compactness, which we again state without proof.

Theorem 2.21. *Let K be compact and suppose that we have a collection of open sets U_j for which $K \subseteq \bigcup_{j\in I} U_j$. Then there exists a* finite subset *$J \subseteq I$ such that $K \subseteq \bigcup_{j\in J} U_j$.*

This is usually said as "every open cover has a finite subcover." In topology, this statement is typically taken as the *definition* of compactness. That this is equivalent to being closed and bounded in $\mathbb{C}$ (or indeed any Euclidean space) is the famous Heine–Borel Theorem. The full details of this, and a thorough discussion of (path-)connectedness, are properly subjects for a course in topology.

[2]Path-connectedness is more useful to us than the related concept of *connectedness*: both convey the idea that a set consists of only one piece (or *component*). For open sets, and for typical domains of functions in this course, connectedness and path-connectedness mean the same thing.

Exercises 2.2 1. Use the ϵ–δ definition (2.10) to prove the following.

(a) $\lim_{z\to z_0} \overline{z} = \overline{z_0}$ (b) $\lim_{z\to 0} \frac{\overline{z}^2}{z} = 0$ (c) $\lim_{z\to 2} \frac{1}{z-i} = \frac{1}{2-i}$ (d) $\lim_{z\to z_0} z^3 = z_0^3$

2. Show that the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ has value 1 at all non-zero points on the real and imaginary axes, but that it has the value -1 at all non-zero points on the line $y = x$. Hence explain why $\lim_{z\to 0} f(z)$ does not exist.

3. Prove part 3(c) of Theorem 2.13.

4. Suppose $\lim_{z\to z_0} f(z) = w_0$. Prove that $\lim_{z\to z_0} |f(z)| = |w_0|$.

5. Use Definition 2.15 to prove part of Theorem 2.17: $\lim_{z\to z_0} f(z) = \infty \implies \lim_{z\to z_0} \frac{1}{f(z)} = 0$.

6. Use Definition 2.15 to prove that

(a) $\lim_{z\to 2i} \frac{iz-1}{z-2i} = \infty$ (b) $\lim_{z\to\infty} \frac{iz-1}{z-2i} = i$.

7. (a) Show that $f(z) = \frac{5iz+1}{3z-2i}$ defines a bijection of the Riemann sphere.
(*Hint: let $w = f(z)$ and solve for z . . .*)

(b) More generally, for any complex numbers $\alpha, \beta, \gamma, \delta$, consider the function $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Prove that this defines a bijection of the Riemann sphere if and only if $\alpha\delta - \beta\gamma \neq 0$. How does this discussion relate to the 2×2 matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$?

8. We complete the proof of Theorem 2.20. Suppose $f : K \to \mathbb{C}$ is continuous.

(a) Let $K \subseteq \mathbb{C}$ be compact. Suppose $(w_n) \subseteq K$ is a sequence such that $(f(w_n))$ is convergent in $\mathbb{C}$. Show that there exists a convergent subsequence (w_{n_k}) and use it to show that $\lim f(w_n) \in f(K)$. Hence conclude that $f(K)$ is closed.

(b) Suppose K is path-connected. If $f(p), f(q) \in f(K)$, show that $\exists w : [0,1] \to f(K)$ such that $w(0) = f(p)$ and $w(1) = f(q)$. Hence conclude that $f(K)$ is path-connected.

2.3 Derivatives & the Cauchy–Riemann Equations

Differentiation is where complex analysis shows significant differences from real analysis, though it won't appears so at first, since the definition of derivative is exactly what you are used to.

Definition 2.22. Let $f : D \to \mathbb{C}$ be a complex function and z_0 an *interior* point of D. We say that f is *differentiable at* z_0 if the following limit exists.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

We call this limit the *derivative* of f at z_0 and denote it by $f'(z_0)$. If f is differentiable everywhere[a] on D then the derivative is a function; $f'(z)$ or $\frac{\mathrm{d}f}{\mathrm{d}z}$.

[a]Such *holomorphic* functions are the main topic of the course: we'll consider them more properly in the next section. Necessarily, D must be an open set if f is to be holomorphic (think about the definition of the limit!).

As with limits, we quickly consider an easy example and record the basic facts.

Example 2.23. The function $f(z) = z^2$ is everywhere differentiable,[a]

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0$$

[a]Remember that when taking limits, we only compute on a punctured disk $0 < |z - z_0| < \delta$, hence the red equality.

The basic rules of differentiation are identical those in real analysis and can be proved similarly. We state them for reference.

Theorem 2.24. *Suppose f and g are differentiable (either at a point z_0 or as functions).*

1. *(Linearity)* *For any constants $a, b \in \mathbb{C}$,* $\frac{\mathrm{d}}{\mathrm{d}z}(af(z) + bg(z)) = af'(z) + bg'(z)$
2. *(Power Law)* *For any $n \in \mathbb{N}_0$,* $\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}$
3. *(Product rule)* $\frac{\mathrm{d}}{\mathrm{d}z}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z)$
4. *(Quotient Rule)* *If $g(z) \neq 0$, then* $\frac{\mathrm{d}}{\mathrm{d}z} \frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$
5. *(Chain Rule)* *If h is differentiable at $g(z_0)$, then $h \circ g$ is differentiable at z_0 and*

$$(h \circ g)'(z_0) = h'\big(g(z_0)\big)g'(z_0)$$

We immediately see that all polynomials and rational function are differentiable. Indeed we can easily compute familiar examples without caring whether we are in $\mathbb{R}$ or $\mathbb{C}$!

Example 2.25. $$\frac{\mathrm{d}}{\mathrm{d}z} \frac{3(z^2 - 2)^5 + z^2}{z^3 + 1} = \frac{[30z(z^2 - 2)^4 + 2z)](z^3 + 1) - 3z^2[3(z^2 - 2)^5 + z^2]}{(z^3 + 1)^2}$$

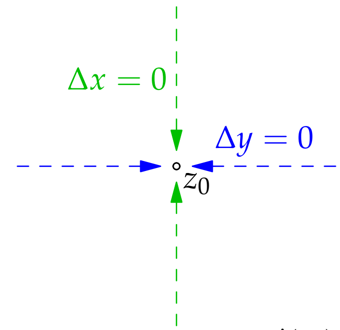
The Cauchy–Riemann Equations

Thus far, differentiation seems uncontroversial. Here is example that might change your mind…

Example 2.26. Consider the complex conjugate function $f(z) = \overline{z}$. We attempt to compute the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y)\to(x_0,y_0)} \frac{x - iy - x_0 + iy_0}{x + iy - x_0 - iy_0} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

For $f'(z)$ to exist, we must obtain the same value *regardless of how* $(\Delta x, \Delta y) \to (0,0)$. There are two obvious ways to take the limit:



Horizontally $\Delta y = 0 \implies \dfrac{f(z) - f(z_0)}{z - z_0} = 1$

Vertically $\Delta x = 0 \implies \dfrac{f(z) - f(z_0)}{z - z_0} = -1$

The quotient takes different values depending on how $(\Delta x, \Delta y) \to (0,0)$. We conclude that $f'(z_0)$ *does not exist*, and that the function $f(z) = \overline{z}$ is not differentiable anywhere!

If the example doesn't surprise you, read it again. There is barely a simpler complex-valued function than the complex conjugate, yet this is not differentiable! We now extend the approach to consider (non-)differentiability more generally.

Let $f(z) = u(x,y) + iv(x,y)$ be written in terms of its real and imaginary parts. As before, we write $z = x + iy$, $z_0 = x_0 + iy_0$ and denote the differences by

$$\Delta z = z - z_0 = \Delta x + i\Delta y = (x - x_0) + i(y - y_0)$$

We attempt to evaluate the limit of the difference quotient

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x,y) - u(x_0,y_0) + i(v(x,y) - v(x_0,y_0))}{\Delta x + i\Delta y}$$

If this exists, then we *must* have the same result when evaluating along straight lines approaching z_0 both horizontally and vertically:

Horizontally We have $\Delta y = 0$ and so the limit becomes

$$\lim_{\Delta x\to 0} \frac{u(x,y_0) - u(x_0,y_0) + i(v(x,y_0) - v(x_0,y_0))}{\Delta x} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x_0,y_0)$$

where we see the *partial derivatives* of the functions u and v.

Vertically We have $\Delta x = 0$, from which

$$\lim_{\Delta y\to 0} \frac{u(x_0,y) - u(x_0,y_0) + i(v(x_0,y) - v(x_0,y_0))}{i\Delta y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)(x_0,y_0)$$

If $f'(z)$ exists, then these limits are equal (to $f'(z)$ itself). We have therefore proved:

Theorem 2.27 (Cauchy–Riemann equations). *If $f(z)$ is complex-differentiable, then the real and imaginary parts of f satisfy the* Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \textit{equivalently} \quad u_x = v_y, \quad u_y = -v_x$$

In such a situation, we can write

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Examples 2.28. 1. $f(z) = \overline{z} = x - iy$ has $u(x,y) = x$ and $v(x,y) = -y$. We quickly see that

$$u_x = 1 \neq -1 = v_y, \qquad u_y = 0 = v_x$$

Since u, v do not satisfy the Cauchy–Riemann equations anywhere (not both of them simultaneously!), f fails to be differentiable anywhere.

2. $f(z) = |z| = \sqrt{x^2+y^2}$ has $u = \sqrt{x^2+y^2}$ and $v = 0$. The Cauchy–Riemann equations are

$$u_x = \frac{x}{\sqrt{x^2+y^2}} = 0 = v_y, \quad u_y = \frac{y}{\sqrt{x^2+y^2}} = 0 = -v_x$$

These equations are satisfied nowhere, and so $f(z)$ is nowhere differentiable.[a]

3. $f(z) = z^2 = x^2 - y^2 + 2ixy$ has $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We check

$$u_x = 2x = v_y, \qquad u_y = -2y = -v_x$$

As expected, u, v satisfy the Cauchy–Riemann equations. Moreover,

$$f'(z) = 2z = 2x + 2iy = u_x + iv_x = v_y - iu_y$$

4. $f(z) = \dfrac{z}{2+|z|^2} = \dfrac{x}{2+x^2+y^2} + \dfrac{iy}{2+x^2+y^2}$. We compute

$$u_x = \frac{2-x^2+y^2}{(2+x^2+y^2)^2} \qquad v_y = \frac{2+x^2-y^2}{(2+x^2+y^2)^2}$$
$$u_y = \frac{-2xy}{(2+x^2+y^2)^2} \qquad -v_x = \frac{2xy}{(2+x^2+y^2)^2}$$

The Cauchy–Riemann equations are satisfied if and only if $xy = 0 = x^2 - y^2$, which is if and only if $x = y = 0$. We conclude that f is not differentiable at any non-zero $z \in \mathbb{C}$.

Note that the Cauchy–Riemann equations only provide a *necessary* condition for differentiability. We do not (yet!) have a sufficient condition. However, we can easily check that this function is differentiable at $z = 0$, straight from the definition of derivative and the continuity of $|z|$:

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{1}{2+|z|^2} = \frac{1}{2}$$

The function f is therefore differentiable at precisely one point.

[a] Note that $u = \sqrt{x^2+y^2}$ isn't even differentiable at $(x,y) = (0,0)$.

A Sufficient Condition for Differentiability?

Since Theorem 2.27 only provides a necessary condition for differentiability, is is most useful in the contrapositive form:

$$\text{Cauchy–Riemann not satisfied} \implies f \text{ not differentiable}$$

We'd like this to be bidirectional, so that the Cauchy–Riemann equations become a sufficient condition for differentiability. This is indeed possible, with one small caveat...

Suppose a complex function $f(z) = u(x,y) + iv(x,y)$ has partial derivatives on an open neighborhood D of a point $z_0 = x_0 + iy_0$. Also suppose f satisfies the Cauchy–Riemann equations and that the partial derivatives $u_x = v_y$ and $u_y = -v_x$ are *continuous* at z_0. The rough idea is to use the linear approximation: for $z \in D$,

$$\begin{aligned} f(z) - f(z_0) &\approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \\ &= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y \\ &= (u_x + iv_x)\Delta x + (-v_x + iu_x)\Delta y \\ &= (u_x + iv_x)(\Delta x + i\Delta y) = (u_x + iv_x)\Delta z \\ \implies \frac{f(z) - f(z_0)}{z - z_0} &\approx u_x(x_0,y_0) + iv_x(x_0,y_0) \end{aligned}$$

The continuity of the partial derivatives at (x_0,y_0) means that the approximation approaches equality as $\Delta z \to 0$. We omit the complete proof since there are too many details. The upshot is a near converse to Theorem 2.27.

Theorem 2.29. *Let u,v have partial derivatives on a neighbourhood of $z_0 = x_0 + iy_0$. Assume $f = u + iv$ satisfies the Cauchy–Riemann equations and has continuous partial derivatives at z_0. Then f is differentiable at z_0 and*

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,v_0)$$

Examples 2.30. 1. Revisiting Example 2.28.4, we see that $f(z) = \frac{z}{2+|z|^2}$ has partial derivatives everywhere; these are continuous and satisfy Cauchy–Riemann at $z = 0$, whence f is differentiable there. Moreover

$$f'(0) = u_x(0,0) + iv_x(0,0) = \frac{1}{2}$$

2. The exponential function is differentiable everywhere: indeed

$$f(z) = e^z = e^x \cos y + ie^x \sin y$$

satisfies

$$u_x = e^x \cos y = v_y, \qquad u_y = -e^x \sin x = -v_x$$

where these are certainly continuous on $\mathbb{C}$. Moreover, as expected,

$$f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$$

Exercises 2.3 1. Use Theorem 2.24 to find the derivatives of the following functions:

(a) $f(z) = \dfrac{1}{z^2+2z}$ (b) $f(z) = (z^3+2iz+1)^7$ (c) $f(z) = \dfrac{(3z^2-i)^3}{(iz^3+4)^2}$

2. Use the limit definition of the derivative to compute the derivative of the functions:

(a) $f(z) = 3z^3 - iz^2$ (b) $f(z) = \dfrac{1}{z^2}$

3. Give a proof of the quotient rule, directly using the definition of the derivative.

4. Use the quotient rule to prove the power law for negative integer exponents: that is

$$\forall n \in \mathbb{N}, \quad \frac{\mathrm{d}}{\mathrm{d}z} z^{-n} = -nz^{-n-1}$$

5. Suppose $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exists, where $g'(z_0) \neq 0$. Use the definition of the derivative to show that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

6. Prove that the functions $f(z) = \operatorname{Re} z$ and $g(z) = \operatorname{Im} z$ are not differentiable anywhere.

7. Exactly as in Example 2.30.2, prove that $\dfrac{\mathrm{d}}{\mathrm{d}z} e^{kz} = ke^{kz}$ for any complex constant k.

8. Consider the Cauchy–Riemann equations for the following functions: what can you conclude, if anything?

(a) $f(z) = \dfrac{1}{\overline{z}-i}$ (b) $f(z) = z^3 - \dfrac{2}{z}$ (c) $f(z) = (|z|^2 + z)^2$

9. Write a complex function $f(z) = f(z, \overline{z})$ as a function of z and $\overline{z}$. For example,

$$f(z) = |z|^2 = z\overline{z}$$

Noting that $x = \frac{1}{2}(z+\overline{z})$ and $y = \frac{1}{2i}(z-\overline{z})$, use the chain rule

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}}$$

to prove that f satisfies the Cauchy–Riemann equations if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$.

Hence give a quick proof that $f(z) = z\overline{z}^2$ is not differentiable when $z \neq 0$.

2.4 Holomorphic and Harmonic Functions

We tend to be most interested in functions which are differentiable on their whole domain.

Definition 2.31. Let $f : D \to \mathbb{C}$ be a function where D is open.

- If f is differentiable at every point of D we say that it is *holomorphic* (or *analytic*[a]) on D.
- We say that f is *holomorphic at* $z_0 \in D$ if it differentiable (holomorphic) on some open set containing z_0.
- An *entire function* is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ (domain $= \mathbb{C}$).

[a] We'll use these terms interchangeably though strictly they have different definitions: analyticity being related to power series representations. A major part of the course involves showing that these definitions are equivalent.

Examples 2.32. 1. The function $f(z) = e^{4z}$ is entire, as is every polynomial.

2. The function $f(z) = \dfrac{iz}{z^2+4}$ is holomorphic on its implied domain $\operatorname{dom} f = \mathbb{C} \setminus \{\pm 2i\}$; indeed, by the quotient rule

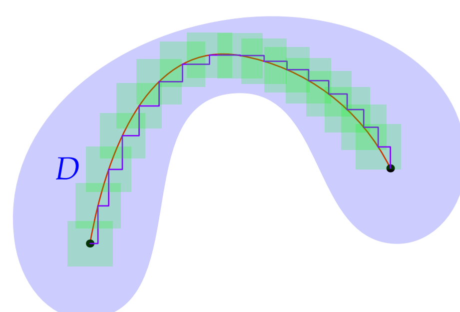
$$f'(z) = \frac{i(z^2+4) - 2iz^2}{(z^2+4)^2} = \frac{i(4-z^2)}{(z^2+4)^2}$$

Our first major result should seem very familiar.

Theorem 2.33. *If $f'(z) = 0$ on a (path-)connected open domain D, then $f(z)$ is constant.*

The set-up relies on Theorems 2.20 and 2.21, though the calculation should be compared to that in real analysis, which also uses the mean value theorem.

Proof. Choose any $p, q \in D$ and join these with a path.
Since D is open, at each point of the path we can choose an open square lying within D centered on the path.
The path is compact, whence it may be covered by finitely many boxes. A zigzag path consisting of finitely many horizontal and vertical segments may now be described within these boxes.
Since $f'(z) = u_x + iv_x = v_y - iu_y = 0$, we see that all four partial derivatives are zero.



For each horizontal segment ($x_1 \le x \le x_2$, y fixed), we apply the mean value theorem to the functions $x \mapsto u(x,y)$ and $x \mapsto v(x,y)$. We therefore have $\widehat{x}, \widetilde{x} \in (x_1, x_2)$ with

$$\frac{u(x_2,y) - u(x_1,y)}{x_2 - x_1} = u_x(\widehat{x}, y) = 0 \quad \text{and} \quad \frac{v(x_2,y) - v(x_1,y)}{x_2 - x_1} = v_x(\widetilde{x}, y) = 0$$

whence u and v, and thus $f(z)$, are constant along any horizontal segment.
The same holds along any vertical segment, this time after utilizing the fact that $u_y = v_y = 0$.
We conclude that $f(q) = f(p)$: since p, q were arbitrary points of D, f is constant. ∎

Corollary 2.34. *If $f(z)$ is holomorphic and $|f(z)|$ is constant, then $f(z)$ is constant.*

This is essentially trivial in the real case: think about why! In the complex case, we need a proof.

Proof. Clearly $|f(z)|^2 = f(z)\overline{f(z)} = k$ is constant. If $k = 0$, we are done. Otherwise, $\overline{f(z)} = \frac{k}{f(z)}$ is holomorphic (quotient rule!). Write $f(z) = u + iv$, whence $\overline{f(z)} = u - iv$, and consider the Cauchy–Riemann equations for *both*:

$$u_x = v_y, \qquad u_y = -v_x, \qquad u_x = -v_y, \qquad u_y = v_x$$

We conclude that all partial derivatives are zero: $f'(z) = 0$ and so f is constant. ∎

We now come to a significant contrast with the real case.

Theorem 2.35. *If $f(z) = u + iv$ is holomorphic, then f is* infinitely differentiable. *Otherwise said:*

- *$f^{(n)}(z)$ exists and is continuous for all $n \in \mathbb{N}$.*
- *u and v have continuous partial derivatives of all orders.*

In real analysis, even functions which are differentiable everywhere need not be *twice* differentiable, let alone infinitely so (Exercise 4). We'll prove this important result later in the course once we've developed some integration theory.

Harmonic Functions We may now consider what happens with the *second* partial derivatives of a holomorphic function:

$$u_{xx} = \frac{\partial}{\partial x}u_x \overset{CR1}{=} \frac{\partial}{\partial x}v_y = v_{yx} \overset{(*)}{=} v_{xy} = \frac{\partial}{\partial y}v_x \overset{CR2}{=} -\frac{\partial}{\partial y}u_y = -u_{yy}$$

Equality of the mixed partial derivatives $(*)$ follows because all derivatives are continuous (Theorem 2.35 and Clairaut's Theorem). The same equation holds for v. We conclude:

Corollary 2.36. *If $f = u + iv$ is holomorphic, then u and v are* harmonic functions*; solutions to* Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Laplace's equation is one of the most important partial differential equations, and is widely used throughout mathematics and physics.

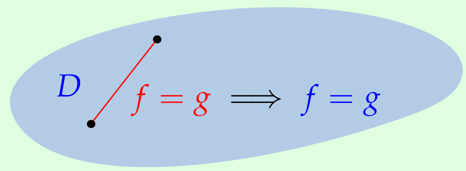
Example 2.37. $f(z) = \frac{1}{z} = \frac{x-iy}{x^2+y^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$; its real and imaginary parts are therefore harmonic away from the origin. Indeed,

$$\begin{aligned} u_{xx} + u_{yy} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{x}{x^2+y^2} = \frac{\partial}{\partial x}\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{\partial}{\partial y}\frac{2xy}{(x^2+y^2)^2} \\ &= \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} - \frac{2x(x^2+y^2) - 8xy^2}{(x^2+y^2)^3} = 0 \end{aligned}$$

Analytic Continuations Though we cannot yet prove it, now is a good time to introduce another surprising property of holomorphic/analytic functions. Since this result will be seen to depend on power series, we'll stick to using the term *analytic* here.

Theorem 2.38. *Suppose that f and g are analytic functions on an open connected domain D and assume that $f(z) = g(z)$ on some path contained in D.*

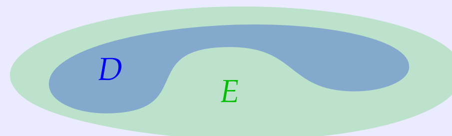
Then $f(z) = g(z)$ throughout D.



This is highly counter-intuitive; you need only know the values of an analytic function on a tiny path to know the full function on its whole (connected) domain! This leads to a new concept.

Definition 2.39. Let $D \subseteq E$ be open connected domains and $g : E \to \mathbb{C}$ an analytic function. Let $f : D \to \mathbb{C}$ be the restriction of g to D; that is $f(z) = g(z)$ on D.

We call g *the analytic continuation* of f to E.



By Theorem 2.38, the analytic continuation of f to E must be unique. There are, however, some subtleties, which we explore a little in the next example:

- Given f analytic on D and $D \subseteq E$, an analytic continuation is not guaranteed to exist.
- The choice of extended domain E *really* matters.

Example 2.40. Consider the principal square root $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ with domain D the first quadrant. In Exercise 10, we'll see that f is differentiable on D.

Consider two analytic continuations of f: in both cases we use the same formula $z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}$ and we distinguish the point $w = e^{-\frac{3\pi i}{4}} = e^{\frac{5\pi i}{4}}$ for comparison.

Let $G = \mathbb{C} \setminus \{-x : x \in \mathbb{R}_0^+\}$ be the plane omitting the *non-positive real axis* and let

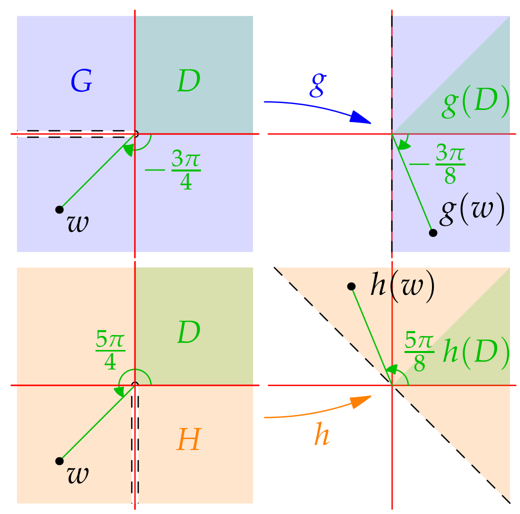
$$g : G \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \operatorname{Arg} z \in (-\pi, \pi)$$

The codomain of g is the right half-plane and $g(w) = e^{-\frac{3\pi i}{8}}$ lies in the *fourth quadrant*.

Let $H = \mathbb{C} \setminus \{-iy : y \in \mathbb{R}_0^+\}$ omit the *non-positive imaginary axis* and let

$$h : H \to \mathbb{C} : z \mapsto \sqrt{r}e^{\frac{i\theta}{2}}, \quad \theta = \arg z \in (-\tfrac{\pi}{2}, \tfrac{3\pi}{2})$$

This time the codomain of h is the upper-right half-plane; moreover $h(w) = e^{\frac{5\pi i}{8}} = -g(w)$ lies in the *second quadrant*.



We have two analytic continuations of f which disagree on the intersection of their domains!

It can moreover be seen that the omissions chosen for G, H are necessary: there is no analytic continuation of f to the punctured plane $\mathbb{C} \setminus \{0\}$, or indeed to any domain in which it is possible to loop completely around the origin. We shall return to this topic later...

Exercises 2.4 1. Suppose $g'(z) = h'(z)$ on an open connected domain D. Prove that $h(z) = g(z) + c$ for some constant $c \in \mathbb{C}$.

(*Equivalently: if $g(z)$ and $h(z)$ are anti-derivatives of $f(z)$ on an open connected domain D then $g(z) - h(z)$ is constant on D.*)

2. Check explicitly that $u = e^{nx} \cos ny$ and $v = e^{nx} \sin ny$ are harmonic functions for any $n \in \mathbb{Z}$.

3. Prove or disprove: if $u, v : D \to \mathbb{R}$ are harmonic functions on an open set $D \subseteq \mathbb{R}^2$, then $f(z) := u(x, y) + iv(x, y)$ is holomorphic.

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Show that f is differentiable, but not *twice* differentiable.

(*Such properties are impossible for complex functions*)

5. Suppose $f(z) = u + iv$ and write $z = x + iy = re^{i\theta}$ in polar form.

(a) Use the chain rule applied to the polar co-ordinate relations

$$x = r\cos\theta, \qquad y = r\sin\theta$$

to compute the partial derivatives u_r, u_θ, v_r and v_θ.

(b) Deduce the polar form of the Cauchy–Riemann equations:

$$ru_r = v_\theta \qquad u_\theta = -rv_r, \qquad f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{z}(u_\theta + iv_\theta)$$

6. Prove the polar form of Laplace's equation:

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$$

7. Show that $u = r^n \cos n\theta$ is a harmonic function for any $n \in \mathbb{N}$: find *two* ways to show that this is true!

8. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(iy) = -iy^3$ whenever $z = iy$ lies on the imaginary axis. What is the value $f(2)$? Explain your answer.

9. The function $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$. Explain why there is no analytic continuation of f that is analytic at $z = 0$.

10. (a) Use Exercise 5 to prove that $f(z) = \sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$ is analytic on the first quadrant, and find $f'(z)$. Moreover, explain why the functions g, h in Example 2.40 are analytic and therefore analytic continuations of f.

(b) Prove that there exists no analytic continuation of g to any set larger than G.
(*Hint: suppose the extended domain contains $-r \in \mathbb{R}^-$. Now use the fact that an analytic continuation must be continuous at $-r \ldots$*)