3 Elementary Functions

We've already considered polynomials, rational functions and, to some extent, n^{th} roots and the exponential. We now develop the logarithmic and trigonometric functions.

3.1 Exponential and Logarithmic Functions

The exponential function $\exp : \mathbb{C} \to \mathbb{C} : z \mapsto e^z$ was defined earlier using Euler's formula

$$\exp(z) = e^z := e^x \cos y + ie^x \sin y \tag{$*$}$$

For reference, we collect some basic properties: at least parts 1–4 should be familiar.

Lemma 3.1. *Throughout let $z, w \in \mathbb{C}$.*

1. *The exponential function is entire and has derivative $\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z$*
2. $e^z \neq 0$
3. $e^{z+w} = e^z e^w$ and $e^{z-w} = \dfrac{e^z}{e^w}$
4. *For all $n \in \mathbb{Z}$, $(e^z)^n = e^{nz}$*
5. e^z *is* periodic *with period $2\pi i$. Moreover,*

$$e^z = e^w \iff z - w = 2\pi in \text{ for some } n \in \mathbb{Z}$$

Sketch Proof.

1. We saw this earlier: check Cauchy–Riemann and compute $\frac{\mathrm{d}}{\mathrm{d}z}e^z = u_x + iv_x$.
2. This follows trivially from ($*$): $e^x > 0$, while $\cos y$ and $\sin y$ are never both zero.
3. Recall the multiple-angle formulæ for cosine and sine.
4. This requires an induction using part 3 with $z = w$.
5. Certainly $e^{w+2\pi in} = e^w$ by the periodicity of sine and cosine. Now suppose $e^z = e^w$ where $z = x + iy$ and $w = u + iv$. By considering the modulus and argument, we see that

$$e^x e^{iy} = e^u e^{iv} \implies \begin{cases} e^x = e^u \\ y = v + 2\pi in \text{ for some } n \in \mathbb{Z} \end{cases}$$

   We conclude that $x = u$ and so $z - w = i(y - v) = 2\pi in$.

∎

Example 3.2. Find all $z \in \mathbb{C}$ such that $e^z = 5(-1+i)$.
Following the Lemma, write $z = x + iy$ and take the polar form of $5(-1+i)$ to see that

$$e^z = 5(-1+i) \iff e^x e^{iy} = 5\sqrt{2}e^{\frac{3\pi i}{4}} \iff \begin{cases} x = \ln(5\sqrt{2}) \\ y = \frac{3\pi}{4} + 2\pi n \quad \text{for some } n \in \mathbb{Z} \end{cases}$$

$$\iff z = \ln(5\sqrt{2}) + \left(\frac{3\pi}{4} + 2\pi n\right) i \quad \text{for some} \quad n \in \mathbb{Z}$$

We see that there are *infinitely many* suitable z!

Duplicate Notation Warning! When $n \in \mathbb{N}$, the expression $e^{\frac{1}{n}}$ can now mean two things. For instance $e^{\frac{1}{3}}$ can mean:

1. The *set* of cube roots of e, namely $\{\sqrt[3]{e}, \sqrt[3]{e}e^{\frac{2\pi i}{3}}, \sqrt[3]{e}e^{-\frac{2\pi i}{3}}\}$;
2. The *real value* $\sqrt[3]{e} \in \mathbb{R}^+$.

Given that e^z is such a common function, in both real and complex analysis, we default to the second meaning: if you mean the set of n^{th} roots, say so! Remember that you can always write $\exp(z)$ for the function if you want to be unambiguous.

The periodicity of the exponential leads to the far more interesting notion of the complex *logarithm.*

Definition 3.3. Let $z = re^{i\theta}$ be a non-zero complex number with principal argument $\theta = \operatorname{Arg} z$. The *principal logarithm* of z is the value

$$\operatorname{Log} z := \ln r + i\theta = \ln|z| + i \operatorname{Arg} z$$

where ln is the usual natural logarithm. The *logarithm* of z is any (and all!) of the values[1]

$$\log z = \ln|z| + i \arg z = \ln r + i(\theta + 2\pi n) : n \in \mathbb{Z}$$

Examples 3.4. 1. Since $-4 = 4e^{\pi i}$, we see that

$$\operatorname{Log}(-4) = \ln 4 + \pi i \quad \text{and} \quad \log(-4) = \ln 4 + (1+2n)\pi i$$

2. Again write in polar form to compute:

$$\operatorname{Log}(\sqrt{3} - i) = \operatorname{Log}(2e^{-\frac{\pi i}{6}}) = \ln 2 - \frac{\pi i}{6} \quad \text{and} \quad \log(\sqrt{3} - i) = \ln 2 - \frac{\pi i}{6} + 2\pi n i$$

These examples involve solving equations of the form $e^w = z$: writing $z = re^{i\theta} = e^{\ln r + i\theta}$ as above, and appealing to part 5 of Lemma 3.1, we instantly see that

$$e^w = z \iff w = \log z$$

Read this carefully, remembering that the logarithm is multi-valued and the exponential periodic:

$$e^{\log z} = z \qquad \text{and} \qquad \log e^w = w + 2\pi n i \quad \text{where } n \in \mathbb{Z}$$

Before moving on, we clear up some of the basic properties of the principal logarithm function. All parts of this should be clear from Definition 3.3.

Lemma 3.5. *Throughout, z and w are complex numbers with $z \neq 0$, and $n \in \mathbb{Z}$.*

- $\operatorname{Log} : \mathbb{C} \setminus \{0\} \to \{w \in \mathbb{C} : \operatorname{Im} w \in (-\pi, \pi]\}$ *is a bijection with inverse* exp.
- $\operatorname{Log} e^w = w + 2\pi n i$ *where $n \in \mathbb{Z}$ is chosen such that* $\operatorname{Im}(\operatorname{Log} e^w) = \operatorname{Im} w + 2\pi n \in (-\pi, \pi]$.
- *If $z \in \mathbb{R}^+$, then* $\operatorname{Log} z = \ln z$ *is the usual* natural *logarithm.*

[1]This is identical to how we use $\arg z$, which, depending on context, means either the *set* $\{\operatorname{Arg} z + 2\pi n i\}$ or some particular value from this set. We'll more formally discuss such *multi-valued* functions in Section 3.2.

The Logarithm Laws Just as the standard rules for exponentiation (Lemma 3.1 parts 3 and 4) apply to the complex exponential, something similar works for the log laws. However, the multi-valued nature of the logarithm makes this a little more subtle.
Suppose non-zero z, w are given: since $|zw| = |z|\,|w|$ and $\arg zw = \arg z + \arg w$, we conclude that

$$\begin{aligned}\log zw &= \ln(|z|\,|w|) + i(\arg z + \arg w) = \ln|z| + i\arg z + \ln|w| + i\arg w\\ &= \log z + \log w\end{aligned}$$

Be very careful with this expression; it is *not* an identity of *functions*. What it really means is the following:

1. We have *set equality*: in particular, the following sets are identical:

$$\begin{aligned}\log z + \log w &= \{\alpha + \beta : \alpha \in \log z, \beta \in \log w\}\\ &= \{|z| + i\operatorname{Arg} z + 2\pi ki + |w| + i\operatorname{Arg} w + 2\pi mi : k, m \in \mathbb{Z}\}\\ \log zw &= \{\ln|zw| + i\operatorname{Arg}(zw) + 2\pi ni : n \in \mathbb{Z}\}\end{aligned}$$

2. There exist *particular choices* of the arguments of z, w and zw so that $\arg zw = \arg z + \arg w$.

Unless you are sure you won't make a mistake, it is therefore safer to write

$$\log zw = \log z + \log w + 2\pi ni \quad \text{for some} \quad n \in \mathbb{Z}$$

Given its restricted range, we can be more precise for the principal logarithm:

$$\operatorname{Log} zw = \operatorname{Log} z + \operatorname{Log} w + 2\pi ni \quad \text{for some} \quad n = 0, -1, 1$$

Example 3.6. Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and $w = \sqrt{2}(1+i) = 2e^{\frac{\pi i}{4}}$. Then

$$\begin{aligned}&\log z = \ln 2 + \frac{5\pi i}{6} + 2\pi ki, \qquad \log w = \ln 2 + \frac{\pi i}{4} + 2\pi mi\\ &\log zw = \log(4e^{\frac{5\pi i}{6} + \frac{\pi i}{4}}) = \log(4e^{\frac{13\pi i}{12}}) = \ln 4 + \frac{13\pi i}{12} + 2\pi ni = \log z + \log w \iff k + m = n\end{aligned}$$

For principal logarithms, however, we need to make a different, explicit, choice:

$$\begin{aligned}&\operatorname{Log} z = \ln 2 + \frac{5\pi i}{6}, \qquad \operatorname{Log} w = \ln 2 + \frac{\pi i}{4}\\ &\operatorname{Log} zw = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \operatorname{Log}(4e^{-\frac{11\pi}{12}}) = \ln 4 - \frac{11\pi i}{12} = \operatorname{Log} z + \operatorname{Log} w - 2\pi i\end{aligned}$$

We can similarly demonstrate the other log law, with the same caveat:

$$\log \frac{z}{w} = \log z - \log w$$

You might assume that the final log law ($\log z^n = n \log z$ whenever $n \in \mathbb{N}$) also makes sense, but you'd be very wrong: an example should explain why you should ignore this law!

Example 3.7. Let $z = -\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}$ and compute what would be meant by the *set* $2\log z$:

$$2\log z = 2\left(\ln 2 + \frac{5\pi i}{6} + 2\pi m i\right) = \ln 4 + \frac{5\pi i}{3} + 4\pi m i$$

This is different from the set

$$\log z^2 = \log(4e^{\frac{10\pi i}{6}}) = \ln 4 + \frac{5\pi i}{3} + 2\pi k i$$

However, in the language of the previous exercise, there would be no problem if each copy of $\log z$ got its own multiples of $2\pi i$: It is therefore safer to state that $\log z^2 \neq 2\log z$.

Since the principal logarithm is a function rather than a set, we can be more precise: for any $n \in \mathbb{N}$,

$$\operatorname{Log} z^n = n \operatorname{Log} z + 2\pi k i \quad \text{for some integer } k \text{ with } |k| \leq \frac{n}{2}$$

Example 3.8. Let $z = e^{-\frac{13\pi i}{16}}$ and consider z^{16}. We see that

$$\operatorname{Log} z^{16} = \operatorname{Log} e^{-13\pi i} = \operatorname{Log} e^{\pi i} = i\pi, \qquad 16 \operatorname{Log} z = -13\pi i \implies \operatorname{Log} z^{16} = 16 \operatorname{Log} z + 14\pi i$$

Exercises 3.1

1. Compute the following:

   (a) $\exp(3 - \frac{\pi}{2} i)$ (b) $\operatorname{Log}(ie)$ (c) $\log(3 - 4i)$ (d) $\operatorname{Log}[(-1+i)^2]$

2. (a) If e^z is real, show that $\operatorname{Im} z = n\pi$ for some integer n.

   (b) If e^z is imaginary, what restriction is placed on z?

3. Show in two ways that the function $f(z) = \exp(z^2)$ is entire, and find its derivative.

4. Prove, for any $z \in \mathbb{C}$, that $\left|\exp(z^2)\right| \leq \exp|z|^2$. What must z satisfy if this is to be *equality*?

5. Find $\operatorname{Re} e^{\frac{1}{z}}$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?

6. Show that $\operatorname{Log} i^3 \neq 3 \operatorname{Log} i$.

7. Show that $\operatorname{Re}(\log(z-1)) = \frac{1}{2}\ln[(x-1)^2 + y^2]$ whenever $z \neq 1$.

8. Prove the above boxed formula for $\operatorname{Log} z^n$.

9. The square roots of i are $\sqrt{i} = e^{\frac{\pi i}{4}}$ and $-\sqrt{i} = e^{-\frac{3\pi i}{4}}$.

   (a) Compute $\operatorname{Log}\sqrt{i}$ and $\operatorname{Log}(-\sqrt{i})$ and check that $\operatorname{Log}\sqrt{i} = \frac{1}{2}\operatorname{Log} i$.

   (b) Show that the set of all logarithms of all square roots of i is

   $$\log i^{\frac{1}{2}} = \left(n + \frac{1}{4}\right)\pi i \quad \text{where} \quad n \in \mathbb{Z}$$

   and therefore deduce that $\log i^{\frac{1}{2}} = \frac{1}{2}\log i$ *as sets*.

3.2 Multi-valued Functions, Branch Cuts and the Power Function

Since each $\log z$ represents a *set* of complex numbers, the complex logarithm is often called a *multi-valued function.* We have already seen several of these beasts:

- The argument of a complex number is any of the values $\arg z = \operatorname{Arg} z + 2\pi n$ where $n \in \mathbb{Z}$. The logarithm is simply a modification of this: $\log z = \ln r + i \arg z$.
- The n^{th} root of z is the set of values $z^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$ where $\omega_n = e^{\frac{2\pi i}{n}}$ is an n^{th} root of unity and $\sqrt[n]{z}$ the principal root.

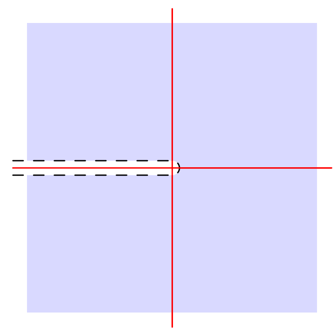
It is an abuse of language to refer to a multi-valued *function,* since a function should assign *exactly one* object to each element of a domain. While this problem can be fixed using equivalence classes, another approach is simpler to visualize.

Definition 3.9. A *branch* of a multi-valued function f is a single-valued function F on a domain D which is *holomorphic* on D and such that each $F(z)$ is one of the values of $f(z)$.

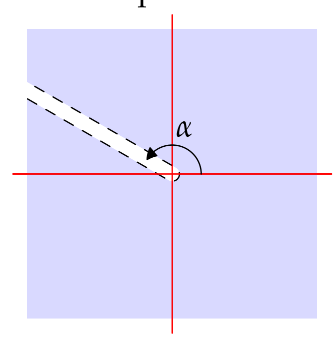
Let $D = \mathbb{C} \setminus \ell$ where ℓ is a line or curve in $\mathbb{C}$. If $F : D \to \mathbb{C}$ is a branch of f, we call ℓ a *branch cut.* A *branch point* is any point common to all branch cuts.

Branches of the Logarithm The *principal branch* of the logarithm is a slightly restricted version of the principal logarithm

$$\operatorname{Log} z = \ln r + i\theta \;\text{ where }\; \theta = \operatorname{Arg} z \in (-\pi, \pi)$$



Principal Branch



α-Branch

The branch cut in this case is the non-positive real axis. To check this, verify the Cauchy–Riemann equations (use polar form[2]):

$$ru_r = r\frac{\partial}{\partial r}\ln r = 1 = \frac{\partial}{\partial \theta}\theta = v_\theta, \qquad u_\theta = 0 = -rv_r$$

The partial derivatives are certainly continuous, whence $\log z$ is holomorphic with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

More generally, for any angle α we could take the branch cut ℓ to be the line with argument α and define a branch of the logarithm by

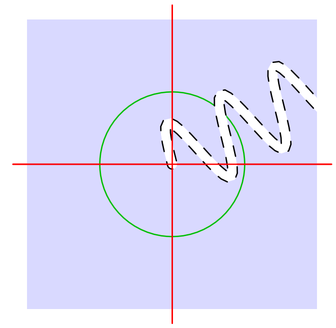
$$\log z = \ln r + i\theta \;\text{ where }\; \theta \in (\alpha, \alpha + 2\pi)$$

The principal branch corresponds to $\alpha = -\pi$. Note that choosing $\alpha = \pi$ produces the *same branch cut* as for $\operatorname{Log} z$, but a *different branch*!

$$\log z = \ln r + i\theta \text{ where } \theta \in (\pi, 3\pi)$$

[2] If you struggle to remember these, compute the partial derivatives of $z = re^{i\theta} = r\cos\theta + ir\sin\theta$!

More esoteric branch cuts are possible. The problem with the logarithm is that if we travel counter-clockwise around the origin, its value increases by $2\pi i$. It is therefore impossible for a branch to be *continuous* (let along *holomorphic*) on any domain D containing such an encircling path; to make $\log z$ single-valued, a branch cut must 'cut' any such path, and must therefore connect the two *branch points* 0 and ∞.



Clarity is crucial here: when you write $\log z$, do you mean a *set*, a particular *element* of that set, or a *branch*? Certain expressions may be true or false depending on the meaning.

Example 3.10. Consider $z = \frac{1}{\sqrt{2}}(1+i) = e^{\frac{\pi i}{4}}$. For the principal branch, we have

$$\operatorname{Log} z^2 = \operatorname{Log} e^{\frac{\pi i}{2}} = \frac{\pi i}{2} = 2 \operatorname{Log} z$$

For the branch with $\alpha = \frac{\pi}{3}$, we have

$$z^2 = e^{\frac{\pi i}{2}} = e^{-\frac{3\pi i}{2}} \implies \log z^2 = -\frac{3\pi i}{2} \neq 2 \log z$$

Recall also (Example 3.7), that *as sets*, $\log z^2 \neq 2 \log z$.

For particular branches of the logarithm, specific versions of the logarithm laws are available. It is not worth trying to remember these; just take care and think out the possibilities!

General Exponential Functions The logarithm can be used to define exponential functions for any non-zero complex base c: choose a value $\log c$ and define

$$c^z := e^{z \log c}$$

Provided c is not a negative real number, the standard is to use the principal logarithm. Regardless of the choice, $\log c$ is constant and the exponential function is holomorphic everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = c^z \log c$$

Example 3.11. Let $c = i = e^{\frac{\pi i}{2}}$ and use the principal logarithm to define

$$i^z := e^{z \operatorname{Log} i} = \exp\left(\frac{\pi i z}{2}\right) = \exp\left(-\frac{\pi}{2} y + i \frac{\pi}{2} x\right) = e^{-\frac{\pi y}{2}} \left[\cos \frac{\pi}{2} x + i \sin \frac{\pi}{2} x\right]$$

It is simple to check the Cauchy–Riemann equations for this function and see that it is holomorphic on $\mathbb{C}$.

If we instead took the α-branch of the logarithm with $\alpha = \frac{\pi}{3}$, then $i = e^{-\frac{3\pi i}{2}}$ and so

$$i^z = \exp\left(\frac{-3\pi i z}{2}\right) = e^{\frac{3\pi y}{2}} \left[\cos \frac{3\pi}{2} x - i \sin \frac{3\pi}{2} x\right]$$

This is still entire, though it is a completely different function! Note that both definitions of i^z agree whenever z is an integer, but for $z = \frac{1}{2}$ these produce the two distinct square roots of i!

Power Functions Similarly to the general exponential function, for any non-zero z and complex number c, we define the (typically multi-valued) function

$$z^c := e^{c \log z}$$

In this case, restricting to the principal branch of the logarithm gives an unambiguous function.

Definition 3.12. The *principal value* of z^c is the function

$$\text{P. V. } z^c := e^{c \operatorname{Log} z}$$

whose domain excludes the non-positive real axis.

Example 3.13. Using the principal branch of the logarithm ($\Theta = \operatorname{Arg} z$), we obtain

$$\text{P. V. } z^{\frac{1}{3}} = \exp\left(\frac{1}{3}(\ln r + i\Theta)\right) = \exp\left(\ln \sqrt[3]{r} + \frac{i\Theta}{3}\right) = \sqrt[3]{r}e^{\frac{i\Theta}{3}} = \sqrt[3]{z}$$

precisely the principal cube-root of z as defined previously.

If we instead choose an α-branch, then $\theta = \arg z \in (\alpha, \alpha + 2\pi)$, from which

$$e^{\frac{1}{3}\log z} = \exp\left(\frac{1}{3}(\ln r + i\theta)\right) = \sqrt[3]{r}\exp\left(\frac{i}{3}(\Theta + (\theta - \Theta))\right) = \sqrt[3]{z}e^{\frac{i(\theta-\Theta)}{3}}$$

Since, for any z, the difference in the arguments $\theta - \Theta = 2\pi n$ is a multiple of 2π, this expression really does return a cube-root of z.

Lemma 3.14. *Choose a branch of the logarithm so that $z^c = e^{c \log z}$ is single-valued. Then z^c is holomorphic on the same domain as the logarithm; moreover $\frac{\mathrm{d}}{\mathrm{d}z} z^c = z^{c-1}$.*

Proof. Since $\log z$ is holomorphic, simply use the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}z}z^c = \frac{\mathrm{d}}{\mathrm{d}z}e^{c\log z} = e^{c\log z}\frac{\mathrm{d}}{\mathrm{d}z}(c\log z) = e^{c\log z}\cdot\frac{c}{z} = ce^{c\log z}e^{-\log z} = ce^{(c-1)\log z} = cz^{c-1}$$

■

Example 3.15. If the principal branch of the logarithm is used, then

$$(zw)^c = \exp(c \operatorname{Log}(zw)) = \exp(c \operatorname{Log} z + c \operatorname{Log} w + 2\pi cni) = z^c w^c e^{2\pi cni}$$

for some $n \in \{0, \pm 1\}$. We do not therefore expect simple exponent rules such as $(ab)^c = a^c b^c$ to hold in complex analysis. Note, however, that this does work in the case where c is an integer.

As an example, again using the principal value, if $z = w = e^{\frac{3\pi i}{4}}$, then $zw = e^{\frac{3\pi i}{2}} = e^{-\frac{\pi i}{2}}$, whence

$$\text{P. V.}(zw)^{5i} = \exp\left(-5i\frac{\pi i}{2}\right) = e^{\frac{5\pi}{2}}$$

$$z^{5i} = \exp\left(5i\frac{3\pi i}{4}\right) = e^{-\frac{15\pi}{4}} \implies z^{5i}w^{5i} = e^{-\frac{15\pi}{2}} = e^{\frac{5\pi}{2}}e^{2\pi\cdot 5i\cdot ni} \text{ with } n = 1$$

Exercises 3.2 1. (a) Show that the function $f(z) = \operatorname{Log}(z - i)$ is holomorphic everywhere except on the portion $x \leq 0$ of the line $y = 1$.

(b) Show that the function $f(z) = \frac{1}{z^2+i} \operatorname{Log}(z + 4)$ is holomorphic everywhere except at the points $\pm\frac{1}{\sqrt{2}(1-i)}$ and on the portion $x \leq -4$ of the real axis.

2. Show that the set $z^{\frac{1}{4}}$ as defined earlier in the course coincides with the set $z^{\frac{1}{4}} := \exp\left(\frac{1}{4} \log z\right)$ as defined in this section.

3. Show that $(1 + i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(i\frac{\ln 2}{2}\right)$ where $n \in \mathbb{Z}$.

4. Find the principal values of the following:

   (a) i^{2i} (b) $(1 - i)^{3i}$ (c) $(-\sqrt{3} + i)^{1+4\pi i}$

5. Suppose c, c_1, c_2 and z are complex numbers where $z \neq 0$. If all the powers involved are principal values, show that,

   (a) $z^{c_1} z^{c_2} = z^{c_1+c_2}$ (b) $(z^c)^n = z^{cn}$ for any $n \in \mathbb{N}$.

6. The power function z^c is *usually* multi-valued. However, if $c = m$ is an integer, prove that z^m is single-valued: i.e. it is independent of the branch of logarithm used in its definition.

7. Check the claim at the bottom of Example 3.11: if $m \in \mathbb{Z}$, then i^m is the same value for the two definitions of i^z.

8. Continuing the previous question, suppose $c \neq 0$ and define $c^z = e^{z \log c}$ where any choice of the branch of the logarithm is made.

   (a) Let $m \in \mathbb{Z}$. Prove that c^m produces the same value, regardless of the branch of logarithm used to define $\log c$.

   (b) If $z = \frac{1}{m}$, show that c^z really is an m^{th} root of c. If the principal branch of the logarithm is used, show that c^z is the principal m^{th} root of c. For every m^{th} root of c, show that there exists a branch of the logarithm for which c^z equals the given m^{th} root.

3.3 Trigonometric and Inverse Trigonometric Functions

It is straightforward to give a sensible definition of the basic trigonometric functions simply by modifying Euler's formula. For instance, if $y \in \mathbb{R}$, then

$$e^{iy} + e^{-iy} = \cos y + i \sin y + \cos y - i \sin y = 2 \cos y$$

This motivates the following.

Definition 3.16. For any $z \in \mathbb{C}$ we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example 3.17. $\cos(\frac{\pi}{4} + i) = \frac{1}{2}(e^{\frac{i\pi}{4} - 1} + e^{\frac{-i\pi}{4} + 1}) = \frac{1}{2}\left(\frac{e^{-1}}{\sqrt{2}}(1+i) + \frac{e}{\sqrt{2}}(1-i)\right) = \frac{e+e^{-1}}{2\sqrt{2}} - i\frac{e-e^{-1}}{2\sqrt{2}}$

Theorem 3.18. *Sine and cosine are entire functions with derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z \qquad \frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z$$

Sine and cosine satisfy the same identities, including double and multiple-angle formulæ, as their real counterparts: for instance

$$\sin^2 z + \cos^2 z = 1, \quad \cos(z+w) = \cos z \cos w - \sin z \sin w, \quad \cos 2z = 2\cos^2 z - 1, \quad \textit{etc.}$$

In particular, $\sin z = \cos(z - \frac{\pi}{2})$ *and* $\cos z = \sin(z + \frac{\pi}{2})$*. Sine and cosine are also* 2π*-periodic and have exactly the same zeros as their real versions:*

$$\sin z = 0 \iff z = n\pi, \quad \cos z = 0 \iff z = \frac{\pi}{2} + n\pi \quad \text{where} \quad n \in \mathbb{Z}$$

The upshot of the Theorem is that sine and cosine behave exactly as you'd expect.
The proofs are straightforward applications of properties of the exponential function. For instance;

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \frac{1}{2i} \frac{\mathrm{d}}{\mathrm{d}z}(e^{iz} - e^{-iz}) = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \cos z$$

and,

$$\sin z = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1 \iff e^{-2y}(\cos 2x + i \sin 2x) = 1 \iff z = \pi n$$

The remaining trigonometric functions are defined in the expected way: for instance,

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \quad \text{whenever} \quad z \neq \frac{\pi}{2} + n\pi$$

All trigonometric functions are holomorphic wherever defined and have the usual expressions for their derivatives: e.g.

$$\frac{\mathrm{d}}{\mathrm{d}z} \tan z = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z$$

Aside: Hyperbolic Functions By considering real and imaginary parts,

$$\begin{aligned}\sin z &= \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = \frac{1}{2i}(e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x)\\ &= \frac{1}{2}(e^{y}+e^{-y})\sin x + \frac{i}{2}(e^{y}-e^{-y})\cos x = \sin x\cosh y + i\cos x\sinh y\\ \cos z &= \cos x\cosh y - i\sin x\sinh y\end{aligned}$$

where $\cosh y = \frac{1}{2}(e^{y}+e^{-y})$ and $\sinh y = \frac{1}{2}(e^{y}-e^{-y})$ are the usual (real) hyperbolic functions. We could have written Example 3.17 this way

$$\cos\left(\frac{\pi}{4}+i\right) = \frac{e+e^{-1}}{2\sqrt{2}} - i\frac{e-e^{-1}}{2\sqrt{2}} = \cos\frac{\pi}{4}\cosh 1 - i\sin\frac{\pi}{4}\sinh 1$$

Hyperbolic functions are a convenient short-cut, but never necessary; use or ignore as you like. All their properties can be derived from their relationship to exponential and trigonometric functions:

$$\cosh z = \frac{e^{z}+e^{-z}}{2} = \cos iz, \qquad \sinh z = \frac{e^{z}-e^{-z}}{2} = -i\sin iz$$

For instance

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}z}\sinh z = -i\frac{\mathrm{d}}{\mathrm{d}z}\sin iz = -i^2\cos iz = \cosh z, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z\\ &\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1\end{aligned}$$

Inverse Trigonometric Functions The standard trigonometric functions can also be inverted. As you might expect, this results in multi-valued functions.

Example 3.19. We find an expression for $\cos^{-1} z$ and compute its derivative.

$$w = \cos^{-1} z \iff z = \cos w \iff 2z = e^{iw} + e^{-iw} \iff (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

which is a quadratic equation in e^{iw}. Applying the quadratic formula, we see that

$$e^{iw} = \frac{2z + (4z^2-4)^{\frac{1}{2}}}{2} = z + i(1-z^2)^{\frac{1}{2}} \iff \cos^{-1} z = w = -i\log\left[z + i(1-z^2)^{\frac{1}{2}}\right]$$

A branch of $\cos^{-1}$ requires a choice of branches of both the square-root *and* the logarithm.
Inverse cosine is holomorphic since it is a composition of holomorphic functions. By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}z}\left[z+i(1-z^2)^{\frac{1}{2}}\right] = \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\left[1 - \frac{iz}{(1-z^2)^{\frac{1}{2}}}\right]\\ &= \frac{-i}{z+i(1-z^2)^{\frac{1}{2}}}\cdot\frac{(1-z^2)^{\frac{1}{2}} - iz}{(1-z^2)^{\frac{1}{2}}} = \frac{-1}{(1-z^2)^{\frac{1}{2}}}\end{aligned}$$

If we fix a branch of the square-root (and logarithm) so that $\cos^{-1} z$ is single-valued, this is necessarily the same branch that appears in the expression for the derivative.

Expressions such as these are not worth memorizing; instead you should become comfortable *deriving* them when needed. Here is the complete list.

Theorem 3.20. *The inverse sine, cosine and tangent functions are given by the expressions*

$$\sin^{-1} z = -i \log\left[iz + (1-z^2)^{1/2}\right] \qquad \cos^{-1} z = -i \log\left[z + i(1-z^2)^{1/2}\right]$$

$$\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$$

Once branches of the square-root and logarithm are chosen, these are holomorphic on their domains and have familiar derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin^{-1} z = \frac{1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1} z = \frac{-1}{(1-z^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z = \frac{1}{1+z^2}$$

The branches of the square-root for the derivatives of inverse sine and cosine are identical to those used in the definitions of the original functions.

Examples 3.21.

1. To evaluate $\sin^{-1}\frac{1}{\sqrt{2}}$ as a complex number, first observe that

$$\sin^{-1}\frac{1}{\sqrt{2}} = -i\log\left[\frac{i}{\sqrt{2}} \pm \sqrt{1-\frac{1}{2}}\right] = -i\log\frac{1}{\sqrt{2}}(i\pm 1)$$

Now evaluate the logarithms separately:

$$-i\log\frac{1}{\sqrt{2}}(i+1) = -i\log e^{\frac{\pi i}{4}} = -i\left[\frac{\pi i}{4} + 2\pi n i\right] = \frac{\pi}{4} + 2\pi n$$

$$-i\log\frac{1}{\sqrt{2}}(i-1) = -i\log e^{\frac{3\pi i}{4}} = -i\left[\frac{3\pi i}{4} + 2\pi n i\right] = \frac{3\pi}{4} + 2\pi n$$

The set of values $\sin^{-1}\frac{1}{\sqrt{2}}$ generated by all branches of the square-root and logarithm is precisely the set we'd have found by computing entirely within $\mathbb{R}$!

2. Of course, we can also evaluate inverse sines that would have no meaning in $\mathbb{R}$. For instance,

$$\begin{aligned}\sin^{-1} 7 &= -i\log[7i \pm \sqrt{-48}] = -i\log(7\pm 4\sqrt{3})i = -i\log(7\pm 4\sqrt{3})e^{\frac{\pi i}{2}} \\ &= -i\left[\ln(7\pm 4\sqrt{3}) + \frac{\pi i}{2} + 2\pi n i\right] = -i\ln(7\pm 4\sqrt{3}) + \frac{\pi}{2} + 2\pi n\end{aligned}$$

Note that $7 > 4\sqrt{3}$, so we are always taking natural log of a positive real number.

3. Compute $\tan^{-1}(i-2\sqrt{3})$. First compute the required fraction in polar form:

$$\frac{i+(i-2\sqrt{3})}{i-(i-2\sqrt{3})} = \frac{2i-2\sqrt{3}}{2\sqrt{3}} = -1 + \frac{i}{\sqrt{3}} = \frac{2}{\sqrt{3}}e^{\frac{5\pi i}{6}}$$

It follows that

$$\tan^{-1}(i-2\sqrt{3}) = \frac{i}{2}\left(\ln\frac{2}{\sqrt{3}} + \frac{5\pi}{6}i - 2\pi n i\right) = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}} + \pi n : \quad n\in\mathbb{Z}$$

Choosing the principal value of the logarithm yields $-\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.

Exercises 3.3 1. Find the real and imaginary parts of $\sin i$, $\cos(1+i)$ and $\tan(2i\ln 5 + \frac{\pi}{2})$.

2. As a sanity check, if $w = -i\log\left[z + (z^2-1)^{\frac{1}{2}}\right]$, compute $\cos w = \frac{1}{2}(e^{iw}+e^{-iw})$ directly and verify that you obtain z, *irrespective* of which branches are chosen.

3. Using the real and imaginary parts of $\sin z$, directly verify that the Cauchy–Riemann equations are satisfied.

4. Prove the following double/multiple-angle formulæ using the definitions in this section:

   (a) $\cos 2z = 2\cos^2 z - 1$

   (b) $\sin(z-w) = \sin z\cos w - \cos z \sin w$

   (c) $\tan(z+w) = \dfrac{\tan z + \tan z}{1 - \tan z \tan w}$

5. Find all the values of $\tan^{-1}(1+i)$.

6. Solve the equation $\cos z = \sqrt{2}$ for z.

7. Recall Exercise 3: check explicitly that $\tan w = i - 2\sqrt{3}$ when $w = -\frac{5\pi}{12} + \frac{i}{2}\ln\frac{2}{\sqrt{3}}$.
   (*Hint: use* $\tan w = \frac{e^{2iw}-1}{i(e^{2iw}+1)}$. *Why…?*)

8. Suppose $z > 1$ is real. Prove that $\operatorname{Re}\sin^{-1} z = \frac{\pi}{2} + 2\pi n$ is independent of z. What is $\operatorname{Im}\sin^{-1} z$.

9. If the same branch of square-root is chosen in each case, prove that $\sin^{-1} z + \cos^{-1} z$ is constant.

10. Derive the expressions for $\tan^{-1} z$ and its derivative in Theorem 3.20.

11. (a) Given that $\cosh z = \cos(-iz)$, find an expression in terms of the complex logarithm for $\cosh^{-1} z$.

    (b) Using your answer to part (a), or otherwise, find all solutions to the equation $\cosh z = \sqrt{3}$.

    (c) Find an expression for the derivative of $\cosh^{-1} z$.