5 Series

The discussion of series in complex analysis differs significantly from the real situation, particularly with regard to two concepts.

- Taylor's Theorem: Holomorphic functions *equal* their Taylor series. This is false in real analysis where differentiable functions need not have, nor equal, a Taylor series.
- Laurent Series: *series* also include negative powers such as $z^{-1} + 3z^{-2} + \cdots$

First we review the basics of sequences and infinite series (of non-negative powers). Hopefully you are familiar with all of this.

5.1 A Brief Review of Sequences and Infinite Series

Post real analysis, there is little specific to say regarding sequences of complex numbers. The notions of limit, convergence and sequential continuity are essentially identical in $\mathbb{C}$ and $\mathbb{R}^2$. For instance:

Definition 5.1. A sequence (z_n) has limit $z \in \mathbb{C}$, written $\lim z_n = z$, if

$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N \implies |z_n - z| < \epsilon$$

Writing $z_n = x_n + iy_n$ and $z = x + iy$ in real and imaginary parts, we see that

$$|z_n - z| \le |x_n - x| + |y_n - y| \le 2\,|z_n - z|$$

from which:

Lemma 5.2. *If $z_n = x_n + iy_n$, then (z_n) converges if and only if both (x_n) and (y_n) converge, in which case*

$$\lim z_n = \lim x_n + i \lim y_n$$

Warning! While this mostly translates to the polar representation $z_n = r_n e^{i\theta_n}$, there is a caveat: the discontinuity of $\operatorname{Arg} z = \Theta$ when z is a non-positive real number means that (Θ_n) need not converge even if (z_n) does.

Example 5.3. The sequence with $z_n = 2i - \frac{1+i}{n}$ has limit $z = 2i$: given $\epsilon > 0$, let $N = \sqrt{2}\epsilon$, then

$$n > N \implies |z_n - z| = \frac{\sqrt{2}}{n} < \frac{\sqrt{2}}{N} = \epsilon$$

The real and imaginary parts are $x_n = -\frac{1}{n}$ and $y_n = 2 - \frac{1}{n}$ which clearly converge to $x = 0$ and $y = 2$ respectively. In polar co-ordinates things are also as expected

$$\lim r_n = \lim \sqrt{\frac{1 + (2n-1)^2}{n^2}} = \lim \frac{2}{n}\sqrt{n^2 - n} = 2$$

$$\lim \Theta_n = \lim \left(\pi \tan^{-1} \frac{(2n-1)/n}{-1/n} \right) = \pi + \tan^{-1}(1 - 2n) = \frac{\pi}{2}$$

Since z_n lies in the second quadrant and $z = 2i$, we never get near the non-negative real axis where Θ_n is discontinuous.

Definition 5.4 (Infinite Series). Given a sequence[a] (z_n), its sequence of *partial sums* (s_n) is

$$s_n = \sum_{k=0}^{n} z_k = z_0 + \cdots + z_n$$

The *infinite series* $\sum z_n := \lim s_n$ is said to converge (diverge) if the sequence (s_n) converges (diverges).
The series *converges absolutely* if $\sum |z_n|$ converges.
The series *converges conditionally* if it converges but not absolutely.

[a]For brevity, assume the initial term is z_0: nothing prevents the initial term being z_{n_0} for any natural number n_0.

Theorem 5.5 (Basic Series Facts). *Let $\sum z_n$ and $\sum w_n$ be series of complex numbers.*

1. *If $z_n = x_n + iy_n$, then $\sum z_n$ converges if and only if $\sum x_n$ and $\sum y_n$ both converge, in which case*

$$\sum z_n = \sum x_n + i \sum y_n$$

2. *If $a \in \mathbb{C}$, and $\sum z_n$ and $\sum w_n$ converge, then $\sum az_n + w_n$ converges, in which case*

$$\sum az_n + w_n = a \sum z_n + \sum w_n$$

3. *(n^{th} term/divergence test)* *If $\sum z_n$ converges, then $\lim z_n = 0$.*
4. *The (real!) comparison, ratio and root tests apply to the series $\sum |z_n|$.*
5. *Absolute convergence implies convergence: moreover $|\sum z_n| \leq \sum |z_n|$.*

Proof. 1. This is immediate from Lemma 5.2.

2, 3. These follow from 1 and the corresponding results for the real series $\sum x_n, \sum y_n$.

4. This requires no proof: $\sum |z_n|$ is a series of non-negative real numbers.
5. Since $\sum |z_n|$ is a convergent series of non-negative terms and $|x_n| \leq |z_n|$, the comparison test proves that $\sum x_n$ is absolutely convergent and thus convergent. Since $\sum y_n$ converges similarly, part 1 shows that $\sum z_n$ converges.
   Finally, apply the triangle inequality $\left|\sum_{k=0}^{m} z_k\right| \leq \sum_{k=0}^{m} |z_k| \leq \sum_{n=0}^{\infty} |z_n|$ and take limits as $m \to \infty$. ■

Exercises 5.1 1. Use the ϵ-N definition to prove that $\lim \frac{2+in}{n} = i$.

2. Give a rigorous proof of 5.2. Sketch a proof of the corresponding statement for the polar representation whenever $\lim z_n$ is non-zero and not a negative real number.
3. Explicitly prove part 2 of Theorem 5.5.
4. Fix $\theta \in (-\pi, \pi]$. Prove that the sequence defined by $z_n = e^{in\theta}$ converges if and only if $\theta = 0$.
5. Use the ϵ-N definition to prove that $\lim \sqrt{i + \frac{1}{n}} = \frac{1+i}{\sqrt{2}} = \sqrt{i}$ where we use the principal value.

5.2 Power Series, Taylor Series and Taylor's Theorem

We first make the identical definition to that in real analysis.

Definition 5.6. A *power series* centered at $z_0 \in \mathbb{C}$ is a function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

A function $f : D \to \mathbb{C}$ is *analytic* if every $z_0 \in D$ has an open neighborhood on which $f(z)$ equals a power series centered at z_0. That is,

$$\forall z_0 \in D, \exists \delta > 0, (a_n) \text{ such that } |z - z_0| < \delta \implies f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

To be analytic at a point z_0 is to be analytic on some open neighborhood of z_0.

The goal of the next two sections is the establishment of a key observation:

A function is analytic if and only if it is holomorphic (differentiable)

Here is the canonical example of a power series.

Example 5.7. (Geometric series) By the n^{th} term test, the power series $\sum z^n$ diverges if $|z| \geq 1$. Otherwise, inside the unit circle $|z| < 1$ we have $z^{n+1} \to 0$, and so

$$s_n - z s_n = 1 - z^{n+1} \implies s_n = \frac{1 - z^{n+1}}{1 - z} \implies \sum_{n=0}^{\infty} z^n = \lim \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}$$

Now let $z_0 \neq 1$; by substituting $z \mapsto \frac{z - z_0}{1 - z_0}$ in the above, observe that

$$\frac{1}{1-z} = \frac{1}{1 - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{1 - z_0}} = \frac{1}{1 - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{1 - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{1}{(1 - z_0)^{n+1}} (z - z_0)^n$$

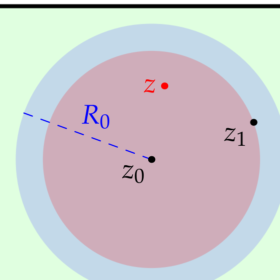
provided $|z - z_0| < |1 - z_0|$. It follows that $f(z) = \frac{1}{1-z}$ is analytic on its domain $\mathbb{C} \setminus \{1\}$.

Note how the geometric series converges on a *disk*. As the next result shows, this is the case for *every* power series, analogous to the concept of the interval/radius of convergence in real analysis.

Theorem 5.8. *Consider a power series $f(z) = \sum a_n (z - z_0)^n$.*

1. *If f converges at a point $z_1 \neq z_0$, then it is absolutely convergent at every point z satisfying $|z - z_0| < |z_1 - z_0|$.*
2. *Define $R_0 := \sup \{|z - z_0| : f(z) \text{ converges}\}$.*

   Then $f(z)$ converges absolutely whenever $|z - z_0| < R_0$ and diverges whenever $|z - z_0| > R_0$.



Proof. 1. By the n^{th} term test, the sequence $(a_n(z_1 - z_0)^n)$ converges (to 0) and is therefore bounded by some $M \in \mathbb{R}^+$. Thus

$$|a_n| |z - z_0|^n = |a_n| |z_1 - z_0|^n \left(\frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq Mr^n \quad \text{where} \quad r = \frac{|z - z_0|}{|z_1 - z_0|} < 1$$

Since $\sum Mr^n$ converges, we conclude (comparison test) that $\sum |a_n| |z - z_0|^n$ converges.

2. By standard properties of the supremum, if $|z - z_0| < R_0$, then $\exists z_1$ such that $f(z_1)$ converges and $|z - z_0| < |z_1 - z_0|$: now apply part (a). The remaining part is an exercise. ■

Definition 5.9. The value R_0 in the theorem is the *radius of convergence* of the power series.

- If $R_0 = \infty$, the series is (absolutely) convergent on $\mathbb{C}$;
- If $R_0 = 0$, the series converges only when $z = z_0$;
- Otherwise, we have a *disk of convergence* with radius $R_0 \in \mathbb{R}^+$ centered at z_0. The *circle of convergence* is the boundary circle of this disk.

As in real analysis:

1. If $R_0 \in \mathbb{R}^+$, we have to test separately for convergence/divergence on the circle of convergence. A key technique for doing this is *Abel's Test* (see Exercise 4).

2. We could use the ratio/root tests to explicitly compute: $R_0 = \liminf |a_n|^{-1/n}$, etc. This is mostly redundant since, as we'll see later, the radius of convergence is usually easier to spot as the distance from z_0 to the nearest point at which f fails to be differentiable.

Before computing further examples, we first revisit a familiar definition and observe a startling difference between the real and complex case.

Definition 5.10. If a function $f(z)$ is infinitely differentiable at z_0, then its *Taylor series* is the power series

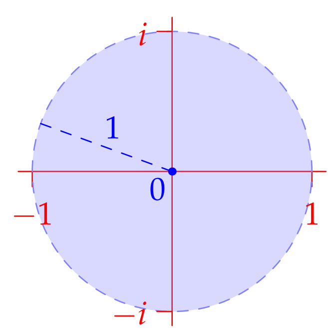
$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

The *Taylor coefficients* are $a_n = \frac{f^{(n)}(z_0)}{n!}$. A *Maclaurin series* is a Taylor series with $z_0 = 0$.

We continue Example 5.7. On the disk $|z| < 1$, the function $f(z) = \frac{1}{1-z}$ has

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} z^n \overset{!!}{=} f(z)$$

The Maclaurin series is therefore the geometric power series representation of $f(z)$ on the open disk $|z| < 1$! In fact this situation is completely general…



Theorem 5.11 (Taylor's Theorem). *If $f(z)$ is holomorphic on a disk $|z - z_0| < R$, then it equals its Taylor series on that disk.*

This is a *very* strong statement in comparison to real analysis, where there exist infinitely differentiable functions which do not equal their Taylor series (see Exercise 5).

Clearly R cannot be larger than the radius of convergence R_0 of the Taylor series. If f is entire, then the result holds for all positive R and the series has radius of convergence is infinity.

Examples 5.12. Further familiar examples translate over from real analysis.

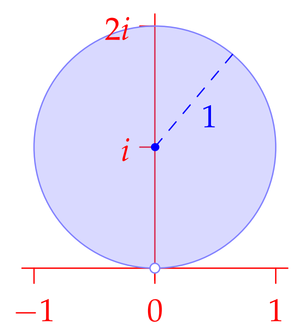
1. Since $f(z) = e^z$ is entire and $f^{(n)}(0) = 1$ for all n, we see that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for all } z \in \mathbb{C}$$

2. $f(z) = \sin z$ is entire with $f^{(2n)}(z) = (-1)^n \sin z$ and $f^{(2n+1)}(z) = (-1)^n \cos z$. Its Maclaurin series is therefore

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \text{for all } z \in \mathbb{C}$$

3. $f(z) = \operatorname{Log} z$ is holomorphic on the open disk $|z - i| < 1$. Whenever $n \geq 1$, we have



$$f^{(n)}(i) = \left. \frac{(-1)^{n-1}(n-1)!}{z^n} \right|_{z=i} = -i^n (n-1)!$$

and we obtain the Taylor series

$$\operatorname{Log} z = \operatorname{Log} i - \sum_{n=1}^{\infty} \frac{i^n}{n}(z-i)^n = \frac{\pi i}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n}$$

Convergence when $|z - i| < 1$ can be directly verified using the comparison test:

$$|z-i| = r < 1 \implies \frac{|z-i|^n}{n} \leq r^n \implies \sum \frac{i^n (z-i)^n}{n} \text{ converges absolutely}$$

At $z = 0$, we recognize the divergent harmonic series $\sum \frac{1}{n}$: the radius of convergence is $R_0 = 1$. Exercise 4 shows that the series converges everywhere else on the boundary circle $|z - i| = 1$.

We'll see the proof in a moment, but first observe that if $f : D \to \mathbb{C}$ is holomorphic, then, for every $z_0 \in D$ it is holomorphic on some disk $|z - z_0| < R$; by Taylor's theorem $f(z)$ equals its Taylor series on this disk and we've therefore proved half of our key observation.

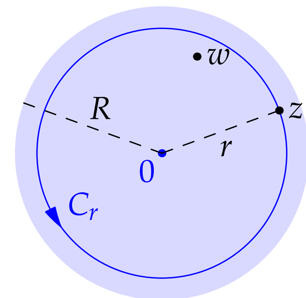
Corollary 5.13. *Every holomorphic function is analytic.*

We'll obtain the converse in the next section.

Why is Taylor's Theorem so much more specific in complex analysis? The answer is that we have available a very powerful tool, namely Cauchy's integral formula.

Proof of Taylor's Theorem. By relabelling $\tilde{f}(z) = f(z - z_0)$, it is enough to prove when $z_0 = 0$, that is for Maclaurin series.

Let w be given where $|w| < R$. By Example 5.7, if $z \neq 0$,



$$\frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k = \frac{1-\left(\frac{w}{z}\right)^n}{z(1-\frac{w}{z})} = \frac{1}{z-w} - \frac{1}{z-w}\left(\frac{w}{z}\right)^n \qquad (*)$$

Choose any circle C_r centered at the origin with radius $r \in (|w|, R)$. Since both 0 and w lie inside C_r, we may apply Cauchy's integral formula *twice*:

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\oint_{C_r}\frac{f(z)}{z-w}\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } w) \\ &= \sum_{k=0}^{n-1}\frac{w^k}{2\pi i}\oint_{C_r}\frac{f(z)}{z^{k+1}}\,\mathrm{d}z + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(subsititute for } \tfrac{1}{z-w} \text{ using } (*)) \\ &= \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k + \frac{w^n}{2\pi i}\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z && \text{(Cauchy for } C_r \text{ around } 0) \end{aligned}$$

All that remains is to control the final integral. Since f is holomorphic on C_r, it is bounded by some $M \in \mathbb{R}^+$. Moreover, for $z \in C_r$ we have $|z - w| \geq= ||z| - |w|| = r - |w|$. Thus

$$\begin{aligned} \left| f(w) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k \right| &= \frac{|w|^n}{2\pi}\left|\oint_{C_r}\frac{f(z)}{z^n(z-w)}\,\mathrm{d}z\right| \\ &\leq \frac{|w|^n M \cdot 2\pi r}{2\pi r^n(r-|w|)} = \frac{Mr}{(r-|w|)}\left(\frac{|w|}{r}\right)^n \end{aligned}$$

This last plainly converges to zero since $|w| < r$. Otherwise said

$$f(w) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}w^k = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n$$

so that $f(z)$ equals its Maclaurin series whenever $|z| < R$. ∎

Exercises 5.2 1. (a) Compute the Maclaurin series of $\cos z$ directly from the definition.

(b) Evaluate the Taylor series of $\sin z$ about $z_0 = \frac{\pi}{2}$ and confirm that it equals your answer to part (a) when z is replaced with $z - \frac{\pi}{2}$.

2. Consider $f(z) = \frac{1}{z}$. For any $z_0 \neq 0$, find the Taylor series of $f(z)$ about z_0. What is its disk of convergence?

3. Finish the proof of Theorem 5.8. Suppose $|z - z_0| > R_0$. Prove that $f(z)$ diverges.

4. The alternating series test was often useful in real analysis to decide convergence at the endpoints of an interval of convergence. Here is a generalization to the complex situation.

   Consider the power series $\sum a_n z^n$ where (a_n) is a *real* sequence such that

$$a_n \geq 0, \quad a_{n+1} \leq a_n, \quad \lim_{n\to\infty} a_n = 0$$

   (a) Write $s_n(z) = \sum_{k=0}^{n} a_k z^k$ for the partial sum and prove that

$$(1 - z)s_n(z) = a_0 - a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1}) z^k$$

   (b) Prove *Abel's Test*: $\sum a_n z^n$ converges everywhere on the *closed* unit disk $|z| \leq 1$, *except perhaps* when $z = 1$.
   (*Hint: show that* $\sum (a_k - a_{k-1}) z^k$ *converges absolutely by comparison with a telescoping series*)

   (c) Verify that $\operatorname{Log} z = \frac{\pi i}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n}$ converges whenever $|z - i| = 1$, except when $z = 0$.

   (d) Prove that the real series $\sum \frac{\cos n\theta}{n}$ converges except when θ is divisible by 2π. For what values of θ does the series $\sum \frac{\sin n\theta}{n}$ converge?

   (e) Find all values of z for which the series $\sum \frac{1+i}{(n+i)(4+3i)^n} (z - 1 + 2i)^n$ converges and sketch the disk of convergence.
   (*Hint: let* $w = \frac{z-1+2i}{4+3i}$)

5. Consider the function

$$f(z) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

   When $z \in \mathbb{R}$ this provides the classic example of an infinitely differentiable function whose Maclaurin series (being identically zero) does not equal the original function except at the origin. When $z \in \mathbb{C}$, why $f(z)$ does not contradict Taylor's Theorem.

6. Why do we need to introduce r in part 1 of the proof of Theorem 5.8? That is, explain why can't we use the comparison test to say

$$|a_n| \, |z - z_0|^n < |a_n| \, |z_1 - z_0|^n \implies \sum |a_n| \, |z - z_0|^n \text{ converges}$$

5.3 Uniform Convergence: Continuity, Integrability and Differentiability

As in real analysis, we want to establish the following useful facts:

1. Representations are unique: if two power series are equal, their coefficients are equal.
2. Power series are continuous, indeed differentiable, inside their disk of convergence.
3. Power series may be differentiated and integrated term-by-term.

The arguments are intertwined. Since these are often similar, even identical, to the real case, we will be brief and postpone all examples until the end. The critical ingredient is uniform convergence.

Definition 5.14. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with n^{th} partial sum $s_n(z)$ and remainder $\rho_n(z) = f(z) - s_n(z)$. We say that the series *converges uniformly* on a domain D if

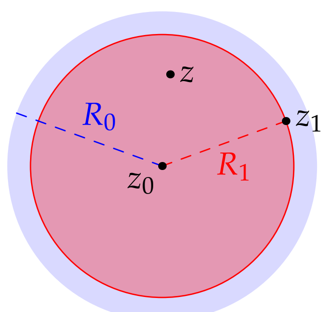
$$\forall \epsilon > 0,\ \exists N \text{ such that } n > N,\ z \in D \implies |\rho_n(z)| < \epsilon$$

Uniformity means that $N = N(\epsilon)$ is independent of the location $z \in D$. If $N = N(\epsilon, z)$ is permitted to depend on z, we'd refer to the convergence as *pointwise.*

Theorem 5.15. *Suppose R_0 is the radius of convergence of a power series about z_0. If $R_1 < R_0$, then the series converges uniformly on the closed disk $|z - z_0| \le R_1$.*

Proof. As preparation, suppose z_1 satisfies $|z_1 - z_0| = R_1$. Since $R_1 < R_0$, the series converges absolutely at z_1 (Theorem 5.8). Denote the n^{th} remainder of this absolutely convergent series by

$$\sigma_n = \sum_{k=n+1}^{\infty} |a_k| \, |z_1 - z_0|^k = \sum_{k=n+1}^{\infty} |a_k| \, R_1^k$$



Now let $\epsilon > 0$ be given. Since the above series converges, we have $\lim_{n\to\infty} \sigma_n = 0$:

$$\exists N \text{ such that } n > N \implies \sigma_n < \epsilon \qquad (*)$$

By the comparison test, if z satisfies $|z - z_0| \le R_1$, then

$$\begin{aligned} |\rho_n(z)| &= \left| \sum_{k=n+1}^{\infty} a_k(z-z_0)^k \right| \le \sum_{k=n+1}^{\infty} |a_k| \, |z-z_0|^k \\ &\le \sum_{k=n+1}^{\infty} |a_k| \, |z_1 - z_0|^k = \sigma_n < \epsilon \end{aligned}$$

∎

Note where the uniformity comes from: we were able to choose N depending only[1] on ϵ, not z.

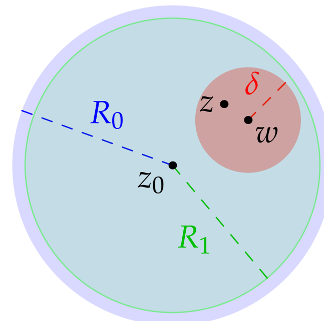
That R_1 is *strictly less* than the radius of convergence R_0 is important. In Exercise 8, we'll see that the convergence of a power series need not be uniform on the full disk of convergence.

[1]It looks as if N might also depend on our choice of z_1 in the first line. However, any suitable z_1 has the same value for $|z_1 - z_0| = R_1$ and thus produces the same sequence (σ_n): it is from the convergence of this sequence that we get N.

Theorem 5.16 (Continuity). *Suppose $f(z) = \sum a_n(z-z_0)^n$ has radius of convergence R_0. Then $f(z)$ is continuous whenever z is interior to the disk of convergence: $|z-z_0| < R_0$.*

This is identical to the famous $\frac{\epsilon}{3}$-proof seen in real analysis.

Proof. Fix w and R_1 such that $|w-z_0| < R_1 < R_0$. Let $\epsilon > 0$ be given. Observe:



- Uniform convergence whenever $|z-z_0| \le R_1$:

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{3} \text{ and } |\rho_n(w)| < \frac{\epsilon}{3}$$

- Openness and continuity (s_n is a polynomial!): for any $n > N$,

$$\exists \delta > 0 \text{ such that } |z-w| < \delta \implies \begin{cases} |z-z_0| < R_1 \\ |s_n(z)-s_n(w)| < \frac{\epsilon}{3} \end{cases}$$

Now put it together to see that $f(z)$ is continuous at w: for any $n > N$,

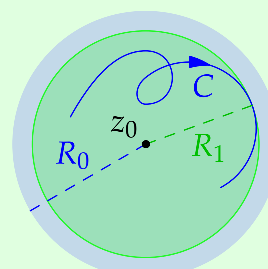
$$\begin{aligned} |z-w| < \delta \implies |f(z)-f(w)| &= |f(z)-s_n(z)+s_n(z)-s_n(w)+s_n(w)-f(w)| \\ &\le |\rho_n(z)| + |s_n(z)-s_n(w)| + |\rho_n(w)| < \epsilon \end{aligned}$$

■

Our treatment now splits from that in real analysis. Since power series are continuous, we may define *contour integrals*. The remaining results follow from a general version of term-by-term integration.

Theorem 5.17. *Let $f(z) = \sum a_n(z-z_0)^n$ have radius of convergence R_0, and C be a contour interior to the disk of convergence: $z \in C \implies |z-z_0| < R_0$. If $g(z)$ is continuous on C, then*

$$\int_C g(z)f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,dz$$



Proof. The integral $\int_C g(z)f(z)\,dz$ exists since f, g are continuous on C. Since C is a compact set:

- C lies inside some closed disk $|z-z_0| \le R_1 < R_0$ on which the series $f(z)$ converges uniformly.
- $g(z)$ is bounded on C by some $M \in \mathbb{R}^+$.

Let C have length L and let $\epsilon > 0$ be given. Since $f(z)$ converges uniformly when $|z-z_0| \le R_1$,

$$\exists N \text{ such that } n > N \implies |\rho_n(z)| < \frac{\epsilon}{ML}$$

Now take integrals and moduli to see that

$$n > N \implies \left| \int_C g(z)f(z)\,dz - \sum_{k=0}^{n} a_k \int_C g(z)(z-z_0)^k\,dz \right| = \left| \int_C g(z)\rho_n(z)\,dz \right| < \epsilon$$

■

Everything we want now follows by choosing specific functions $g(z)$.

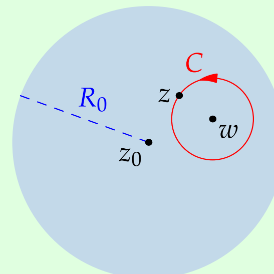
Corollary 5.18. *Suppose $f(z) = \sum a_n(z-z_0)^n$ is a power series with radius of convergence $R_0 > 0$.*

1. *(Term-by-term integration) Let $g(z) = 1$ to see that*

$$\int_C f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}\Big|_{C(start)}^{C(end)}$$

2. *(Holomorphicity) $\oint_C f(z)\,dz = 0$ for every simple closed contour, whence $f(z)$ is holomorphic inside the circle of convergence. In particular, every analytic function is holomorphic.*

3. *(Term-by-term differentiation) Given $|w - z_0| < R_0$, let $g(z) = \frac{1}{2\pi i(z-w)^2}$ and apply Cauchy's integral formula on a small circle around w:*



$$f'(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-w)^2}\,dz = \sum \frac{a_n}{2\pi i}\oint \frac{(z-z_0)^n}{(z-w)^2}\,dz$$
$$= \sum a_n \frac{d}{dz}\Big|_{z=w}(z-z_0)^n = \sum a_n n(z-z_0)^{n-1}$$

4. *(Unique representation) The power series is the Taylor series of $f(z)$: that is, $a_n = \frac{f^{(n)}(z_0)}{n!}$.*

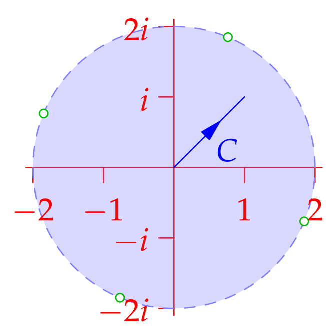
Exercise 6 considers part 4 and its implications. Since analytic and holomorphic are now equivalent, we'll retire the latter for the rest of the course.

Examples 5.19. By uniqueness of representation, we can compute Taylor/Maclaurin series *algebraically*: if a function equals a series, that's the one we want regardless of how we found it!

1. $f(z) = z^3e^{z^2} = z^3\sum_{n=0}^{\infty}\frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{2n+3}}{n!}$ is the Maclaurin series of $f(z)$. Since the radius of convergence is infinite, the function equals its Maclaurin series everywhere on $\mathbb{C}$.

2. The function $f(z) = \begin{cases}\frac{\sin z}{z} & \text{if } z \neq 0\\ 1 & \text{if } z = 0\end{cases}$ is entire since it equals the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}$.

3. We find the Maclaurin series of $f(z) = \frac{1}{z^4+16i}$ algebraically:



$$f(z) = \frac{1}{16i\left(1-\frac{z^4}{-16i}\right)} = \frac{1}{16i}\sum_{n=0}^{\infty}\left(\frac{z^4}{-16i}\right)^n = \sum_{n=0}^{\infty}\frac{i^{n-1}}{16^{n+1}}z^{4n}$$

This converges whenever $\left|\frac{z^4}{-16i}\right| < 1 \iff |z| < 2$, equalling the distance from the center to the nearest point(s) that $f(z)$ fails to be analytic. If C is the straight line from $z = 0$ to $z = 1+i$, then

$$\int_C f(z)\,dz = \sum_{n=0}^{\infty}\frac{i^{n-1}}{16^{n+1}}\int_C z^{4n}\,dz = \sum_{n=0}^{\infty}\frac{i^{n-1}(1+i)^{4n+1}}{16^{n+1}(4n+1)} = \sum_{n=0}^{\infty}\frac{1-i}{16(4n+1)}\left(\frac{-i}{4}\right)^n$$

Exercises 5.3 1. Find a power series representation and the radius of convergence:

(a) $f(z) = \dfrac{z}{4-z}$ about $z_0 = 0$;

(b) $f(z) = z\sin z^2$ about $z_0 = 0$;

(c) $f(z) = \cosh 3z$ about $z_0 = \dfrac{i\pi}{9}$

2. *Without computing derivatives*, find the Taylor series for $f(z) = \frac{1}{z}$ about $z_0 \neq 0$. By differentiating term-by-term, find the Taylor series of $\frac{1}{z^2}$ about z_0.

3. By expressing it as a Maclaurin series, show that the following function is entire:

$$f(z) = \begin{cases} \frac{1}{z^2}(1-\cos z) & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{cases}$$

4. (a) By integrating the Taylor series for z^{-1} about $z_0 = 1$, prove that

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \quad \text{whenever } |z-1| < 1$$

(b) Prove that the following function is analytic on the domain $0 < |z|$, $\operatorname{Arg} z \in (-\pi, \pi)$:

$$f(z) = \begin{cases} \frac{\operatorname{Log} z}{z-1} & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

5. Consider the Maclaurin series $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ on the disk $|z| < 1$. Show that $h(z) = \frac{1}{z^2+1}$ is the analytic continuation of $f(z)$ to $\mathbb{C} \setminus \{i, -i\}$.

6. (a) Prove part 4 of Corollary 5.18: if $f(z) = \sum a_n(z-z_0)^n$, prove that $f^{(m)}(z_0) = m!a_m$ so that the series really is the Taylor series of $f(z)$.
(*Hint: let* $g(z) = \frac{m!}{2\pi i(z-z_0)^{m+1}}$ *in Theorem 5.17*)

(b) Explain carefully why every power series defines an analytic function.
(*Think carefully about the definitions and what we've proved in the last two sections!*)

7. Suppose that the series $\sum a_n(z-z_0)^n$ has radius of convergence R_0 and that $f(z) = \sum a_n(z-z_0)^n$ whenever $|z-z_0| < R_0$. Prove that

$$R_0 = \inf\{|\hat{z} - z_0| : f(z) \text{ non-analytic or undefined at } \hat{z}\}$$

(*R_0 is essentially the distance from z_0 to the nearest point at which $f(z)$ is non-analytic*)

8. (Hard) Consider $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ on $|z| < 1$.

(a) Let $R_1 < 1$. Explicitly check uniform convergence when $|z| \leq R_1$. That is, given $\epsilon > 0$, find an explicit N such that

$$n > N \implies |\rho_n(z)| = \left| f(z) - \sum_{k=0}^{n} z^k \right| < \epsilon \text{ whenever } |z| \leq R_1$$

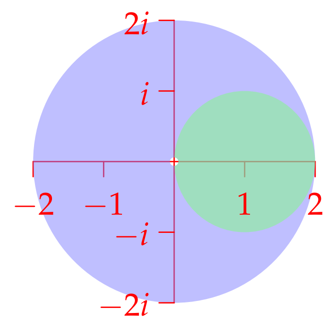
(b) Prove that $f(z)$ is *not* uniformly convergent on $|z| < 1$.
(*Hint: Let $\epsilon = 1$ and try to get a contradiction...*)

5.4 Laurent Series

While Taylor series are undeniably useful, they also have key weaknesses, particularly with regard to their domains being *disks*. We motivate a more general construction with an example.

Example 5.20. $f(z) = \frac{1}{z(2-z)}$ can be written as a Taylor series centered at $z = 1$:

$$f(z) = \frac{1}{1-(z-1)^2} = \sum_{n=0}^{\infty} (z-1)^{2n} \quad \text{whenever } |z-1| < 1$$



However, the most interesting aspects of $f(z)$ involve its behavior near the points $z = 0, 2$. Because of their disk-domains, we can't use Taylor series to loop around these points.
As an alternative, expand $\frac{1}{2-z}$ in a power series centered at 0:

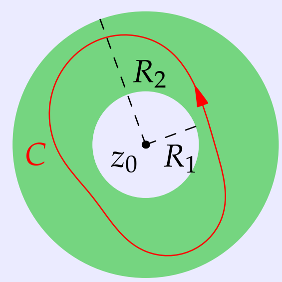
$$f(z) = \frac{1}{2z(1-\frac{z}{2})} = \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=-1}^{\infty} \frac{z^n}{2^{n+2}} = \frac{1}{2z} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \cdots$$

By construction, this second series is valid on the punctured disk $0 < |z| < 2$. The larger domain, particularly the fact that it encircles the origin, provides an obvious advantage over the Taylor series.

Definition 5.21. Let $R_1 < R_2$ and suppose $f(z)$ is analytic on the *annulus* $R_1 < |z - z_0| < R_2$. Its *Laurent series* about z_0 is the expression[a]

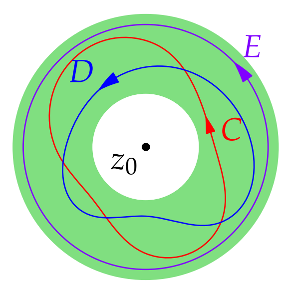
$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

and C is a simple closed contour encircling z_0 within the annulus.



[a]If you prefer, write $\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ where $b_n = \frac{1}{2\pi i} \oint_C (z-z_0)^{n-1} f(z) \, dz$.

- As in Example 5.20, the inner radius can be $R_1 = 0$ and the domain a punctured disk. As with Taylor series, the outer radius can be infinite.
- The coefficients a_n are independent of the choice of contour C.
  To see this, suppose D is another simple closed curve encircling z_0, and choose a circle E outside both C and D. Since $\frac{f(z)}{(z-z_0)^{n+1}}$ is analytic on the annulus, two applications of Cauchy–Goursat yield

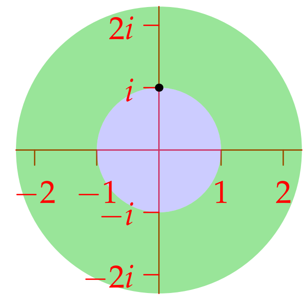


$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \oint_E \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \oint_D \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

- If $f(z)$ is analytic on the *disk* $|z - z_0| < R_2$, then the Laurent series equals the Taylor series:
  - $n \geq 0 \implies a_n = \frac{f^{(n)}(z_0)}{n!}$ by Cauchy's integral formula;
  - $n < 0 \implies a_n = 0$ by Cauchy–Goursat.

It is usually difficult to compute a Laurent series directly using the definition, since it requires infinitely many contour integrals! Thankfully, as we'll see shortly, all the standard facts regarding Taylor series translate to this new situation. In particular, if $f(z) = \sum a_n(z-z_0)^n$, then the series is the Laurent series of $f(z)$ (Corollary 5.26). This makes computing examples much easier!

Examples 5.22. 1. Whenever $|z| < 1$ we have the Taylor series

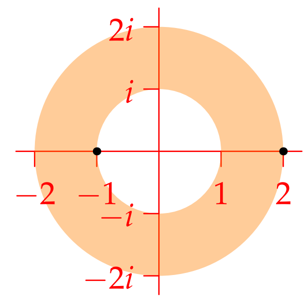


$$\frac{1}{z-i} = \frac{1}{-i(1-\frac{z}{i})} = i\sum_{n=0}^{\infty}(-iz)^n = i + z - iz^2 - z^3 + iz^4 + \cdots$$

When $|z| > 1$, we have the Laurent series

$$\frac{1}{z-i} = \frac{z}{(1-\frac{i}{z})} = \sum_{n=0}^{\infty} i^n z^{-n-1} = \frac{i}{z} - \frac{1}{z^2} - \frac{i}{z^3} + \frac{1}{z^4} + \cdots$$

2. Whenever $1 < |z| < 2$ we have a Laurent series



$$\begin{aligned}\frac{3}{(2-z)(1+z)} &= \frac{1}{2-z} + \frac{1}{1+z} = \frac{1}{2(1-\frac{z}{2})} + \frac{1}{z(1+\frac{1}{z})}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \frac{1}{z}\sum_{m=0}^{\infty}(-z)^{-m}\\ &= \cdots + z^{-3} - z^{-2} + z^{-1} + \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots\end{aligned}$$

3. Since e^z has Maclaurin series $\sum \frac{z^n}{n!}$ valid on the entire complex plane, we obtain the Laurent series expansion

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

on the punctured plane $z \neq 0$. Explicitly evaluating the integrals $a_n = \frac{1}{2\pi i}\oint_C \frac{e^{1/z}}{z^{n+1}}\,\mathrm{d}z$ would be extremely irritating!

4. Again using Maclaurin series, we obtain another Laurent series valid on the punctured plane $z \neq 0$:

$$\frac{1}{z^7}\sin z^2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{4n-5} = z^{-5} - \frac{1}{6}z^{-1} + \frac{1}{120}z^3 - \frac{1}{5040}z^7 + \cdots$$

5. Multiplying term-by-term, and since we need *both* Maclaurin series to be valid, we obtain a Laurent series valid on the punctured disk $0 < |z| < 1$:

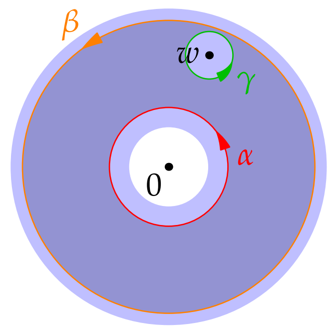
$$\begin{aligned}\frac{1}{z(z-1)(z-2i)} &= \frac{1}{z}\left(\sum_{n=0}^{\infty}(-1)^n z^n\right)\left(\sum_{m=0}^{\infty}\left(\frac{i}{2}\right)^m z^m\right)\\ &= \frac{1}{z}\left(1 - z + z^2 - z^3 + \cdots\right)\left(1 + \frac{i}{2}z - \frac{1}{4}z^2 - \frac{i}{8}z^3 + \cdots\right)\\ &= \frac{1}{z} + \left(-1 + \frac{i}{2}\right) + \left(\frac{3}{4} - \frac{i}{2}\right)z + \left(-\frac{3}{4} + \frac{3i}{8}\right)z^2 + \cdots\end{aligned}$$

Theory time!

Having seen a few examples, we should properly state and prove the main properties of Laurent series. These are very similar to the corresponding arguments for Taylor series; mostly it is an issue of keeping track of two series at once.

Theorem 5.23 (Laurent's Theorem). *An analytic function on an open annulus equals its Laurent series.*

Proof. By a simple translation, it is enough to prove when $z_0 = 0$. Let w in the annulus be given.



Since the annulus is open, we may choose three non-overlapping circles α, β, γ with radii $R_\alpha, R_\beta, R_\gamma$ as in the picture:

- γ a small circle centered at w inside the annulus;
- α, β centered at 0, α inside and β outside w.

Since $\frac{f(z)}{z-w}$ is analytic on the region inside β with interior boundaries α and γ, Cauchy–Goursat says that

$$\left(\oint_\beta - \oint_\alpha - \oint_\gamma\right)\frac{f(z)}{z-w}\,dz = 0 \implies f(w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,dz$$

As in the proof of Taylor's theorem, we expand

$$\frac{1}{z-w} = \frac{1}{z}\sum_{k=0}^{n-1}\left(\frac{w}{z}\right)^k + \frac{1}{z-w}\left(\frac{w}{z}\right)^n = -\frac{1}{w}\sum_{k=1}^{n}\left(\frac{z}{w}\right)^{k-1} + \frac{1}{z-w}\left(\frac{z}{w}\right)^n$$

and use this to attack the two integrals:

$$\frac{1}{2\pi i}\oint_\beta \frac{f(z)}{z-w}\,dz = \sum_{k=0}^{n-1}\underbrace{\frac{w^k}{2\pi i}\oint_\beta \frac{f(z)}{z^{k+1}}\,dz}_{a_k w^k} + \frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,dz$$

$$\frac{-1}{2\pi i}\oint_\alpha \frac{f(z)}{z-w}\,dz = \sum_{k=1}^{n}\underbrace{\frac{1}{2\pi i w^k}\oint_\alpha z^{k-1}f(z)\,dz}_{a_{-k}w^{-k}} - \frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,dz$$

Since $f(z)$ is continuous on the closed bounded annulus between α, β, it has an upper bound M. Moreover, whenever $z \in \alpha \cup \beta$, we have $|z-w| > R_\gamma$. The triangle inequality finishes things off:

$$\begin{aligned}\left|f(w) - \sum_{k=-n}^{n-1} a_k w^k\right| &= \left|\frac{1}{2\pi i}\left(\oint_\beta - \oint_\alpha\right)\frac{f(z)}{z-w}\,dz - \sum_{k=-n}^{n-1} a_k w^k\right| \\ &\le \left|\frac{w^n}{2\pi i}\oint_\beta \frac{f(z)}{z^n(z-w)}\,dz\right| + \left|\frac{1}{2\pi i w^n}\oint_\alpha \frac{z^n f(z)}{z-w}\,dz\right| \\ &\le \frac{MR_\beta}{R_\gamma}\left(\frac{|w|}{R_\beta}\right)^n + \frac{MR_\alpha}{R_\gamma}\left(\frac{R_\alpha}{|w|}\right)^n \xrightarrow[n\to\infty]{} 0\end{aligned}$$

■

By substituting $w = (z - z_0)^{-1}$ in a series of negative powers

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} a_{-n} w^n$$

and applying Theorems 5.8, 5.15 and 5.16 to the power series in w, we may conclude:

Corollary 5.24. *Given a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, define*

$$R_1 = \inf\{|z - z_0| : f(z) \text{ converges}\}, \qquad R_2 = \sup\{|z - z_0| : f(z) \text{ converges}\}$$

Then:

1. *The series converges absolutely on the annulus $R_1 < |z - z_0| < R_2$ to a continuous function.*
2. *The convergence is uniform on any closed sub-annulus.*

Definition 5.25. The annulus $R_1 < |z - z_0| < R_2$ is the (open) *annulus of convergence* of the Laurent series. As with power series, convergence on the boundary circles must be checked separately.

We also obtain the analogues of Theorem 5.17 and Corollary 5.18: some details are in the exercises.

Corollary 5.26. *Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ has annulus of convergence $R_1 < |z - z_0| < R_2$.*

1. *(Term-by-term Integration) If $g(z)$ is continuous on a contour C lying inside the annulus, then*

$$\int_C g(z) f(z) \, dz = \sum_{n=-\infty}^{\infty} a_n \int_C g(z)(z - z_0)^n \, dz$$

   In particular, $f(z)$ may be integrated term-by-term along C.

2. *(Analyticity/Derivatives) $f(z)$ is analytic on the annulus and $f'(z) = \sum_{n=-\infty}^{\infty} a_n n (z - z_0)^{n-1}$*
3. *(Uniqueness) $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is the Laurent series of $f(z)$.*

Now all the abstraction is out of the way, we can more easily compute Laurent series and Examples 5.22 are all valid. Here are a couple more.

Examples 5.27. 1. In accordance with part 2 of Corollary 5.26,

$$\frac{d}{dz} e^{1/z} = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=1}^{\infty} \frac{-z^{-1-n}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=1}^{\infty} \frac{z^{-(n-1)}}{(n-1)!} = -\frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = -\frac{1}{z^2} e^{1/z}$$

2. To compute the integral $\oint_C \frac{1}{z^5} \sin z^2 \, dz$ on a simple closed contour encircling the origin, we use the Laurent series and observe that all but one of the integrals evaluates to zero:

$$\oint_C \frac{1}{z^7} \sin z^2 \, dz = \sum_{n=0}^{\infty} \oint_C \frac{(-1)^n}{(2n+1)!} z^{4n-5} \, dz = \oint_C \frac{(-1)^1}{(2+1)!} z^{4-5} = -\frac{1}{3}\pi i$$

Exercises 5.4 1. Using Definition 5.21, directly compute the Laurent series of $f(z) = \frac{1}{z(2-z)}$ on the punctured disk $0 < |z| < 2$ and verify that you obtain the series in Example 5.20.

2. Find a Laurent series representation for each function. Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

   (a) $f(z) = \frac{3}{z^2}e^{2z}$ whenever $|z| > 0$;

   (b) $f(z) = \cos\frac{i}{z}$ whenever $|z| > 0$;

   (c) $f(z) = \frac{1}{1+z^3}$ when $1 < |z|$ (*Hint: let* $w = z^{-1}$).

3. On each domain, find a Laurent series about $z_0 = 0$ for the function

$$f(z) = \frac{1}{z(z-2i)} = \frac{i}{2}\left(\frac{1}{z} - \frac{1}{z-2i}\right)$$

   (a) $D_1 = \{z : 0 < |z| < 2\}$;

   (b) $D_2 = \{z : |z| > 2\}$ (*again let* $w = z^{-1}$).

4. Repeat the previous question for

$$f(z) = \frac{1-2i}{(z-1)(z-2i)} = \frac{1}{z-1} - \frac{1}{z-2i}$$

   Also find $\oint_C f(z)\,dz$ where C is a simple closed curve in the given domain encircling the origin.

   (a) $D_1 = \{z : 0 < |z| < 1\}$ (*this is a Taylor series*);

   (b) $D_2 = \{z : 1 < |z| < 2\}$;

   (c) $D_3 = \{z : |z| > 2\}$.

5. Show that when $0 < |z-1| < 2$, we have

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

6. Let a be complex number. Show that

$$\frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad \text{whenever } |a| < |z|$$

7. Suppose $f(z) = \sum a_n(z-z_0)^n$ is a series satisfying the hypotheses of Corollary 5.26.

   (a) Suppose part 1 has been proved. Explain why the function $f(z) - a_{-1}(z-z_0)^{-1}$ is analytic on the annulus. Hence conclude that $f(z)$ is analytic on the annulus.
   (*This is different to Corollary 5.18 since $a_{-1}(z-z_0)^{-1}$ has no anti-derivative on the annulus!*)

   (b) In order to mimic the proof of Corollary 5.18 to show that $f(z)$ is differentiable term-by-term, what properties must the curve C have?

   (c) Prove part 3 (*recall Exercise 5.3.6 - the same hint works!*).