6 Residues and Poles

6.1 Residues and Cauchy's Residue Theorem

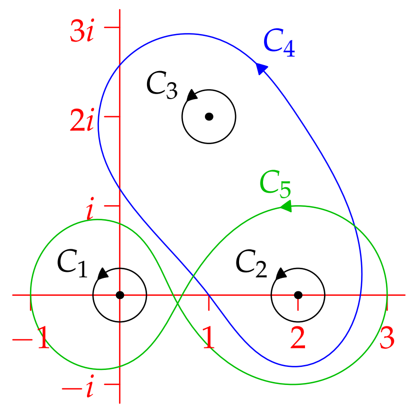
The goal of this section is the efficient computation of contour integrals of analytic functions. Essentially everything will depend on two crucial facts:

- The Cauchy–Goursat Theorem (f analytic on and inside $C \implies \oint_C f = 0$), and its extension to a region with finitely many interior boundary curves.
- If C encircles z_0, then $\oint_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$

Make sure you are familiar with these statements before proceeding!

Example 6.1. Consider the function

$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$$



which is analytic except at the points $z_1 = 0$, $z_2 = 2$, $z_3 = 1 + 2i$. Several curves are drawn. The integral round the small circle C_1 should be clear from the 'crucial' facts:

$$\oint_{C_1} f(z) \, dz = 3 \oint_{C_1} \frac{dz}{z} + \oint_{C_1} \frac{dz}{z^2} + \oint_{C_1} \frac{5i \, dz}{z-2} + \oint_{C_1} \frac{dz}{z-1-2i}$$
$$= 3 \cdot 2\pi i + 0 + 0 + 0 = 6\pi i$$

since the latter three integrands are analytic on and inside C_1. Similarly,

$$\oint_{C_2} f(z) \, dz = 5i \oint_{C_2} \frac{dz}{z-2} = -10\pi, \qquad \oint_{C_3} f(z) \, dz = \oint_{C_3} \frac{dz}{z-1-2i} = 2\pi i$$

More interesting are the curves C_4 and C_5. Since $f(z)$ is analytic on and between C_4 and C_2/C_3, Cauchy–Goursat tells us that

$$\oint_{C_4} f(z) \, dz = \oint_{C_2} f(z) \, dz + \oint_{C_3} f(z) \, dz = 2\pi(i - 5)$$

C_5 appears a little trickier, though it becomes easy once you visualize it as *two* contours: the first encircles z_2 *counter-clockwise* while the second passes *clockwise* around z_1. We conclude that

$$\int_{C_5} f(z) \, dz = \oint_{C_2} f(z) \, dz - \oint_{C_1} f(z) \, dz = -2\pi(5 + 3i)$$

The example suggests that the value of any integral round a simple closed contour can be evaluated as a linear combination

$$\int_C f(z) \, dz = \lambda_1 \oint_{C_1} f + \lambda_2 \oint_{C_2} f + \lambda_3 \oint_{C_3} f$$

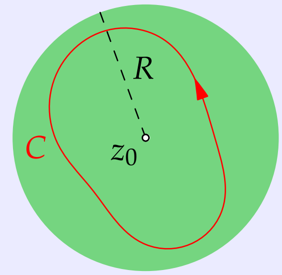
where λ_k denotes the number of times C orbits z_k in a counter-clockwise direction.

To properly develop this idea, we need a little formality.

Isolated Singularities and their Types

Definition 6.2. Suppose $f(z)$ is analytic on an punctured disk $0 < |z - z_0| < R$ of a point z_0, but not at z_0 itself. We call z_0 an *isolated singularity* of $f(z)$.

By Laurent's Theorem, $f(z)$ equals its Laurent series on this domain:



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

and C is any simple closed contour encircling z_0. The *residue* of $f(z)$ at z_0 is the coefficient

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \, \mathrm{d}z$$

The type of isolated singularity is determined by the structure of the Laurent series:

Removable Singularity The Laurent series is a Taylor series. There are no negative powers; the series, and $f(z)$, may be extended analytically to z_0.

Pole of order m The highest negative power in the Laurent series is $(z - z_0)^{-m}$. A pole of order 1 is typically called a *simple pole.*

Essential Singularity The Laurent series has infinitely many many non-zero negative terms.

Examples 6.3.

1. The series $f(z) = \sum_{n=0}^{\infty} 3^{-n} (z - 2i)^n$ defined on the punctured disk $0 < |z - 2i| < 3$ has a removable singularity at $z_0 = 2i$ with residue $\operatorname*{Res}_{z=2i} f(z) = 0$. Indeed the function is a geometric series and thus equals

$$f(z) = \frac{1}{1 - \frac{z-2i}{3}} = \frac{3}{3 + 2i - z}$$

on the punctured disk. Certainly this extends analytically to $f(2i) = 1$.

2. (Example 6.1) The function $f(z) = \frac{3}{z} + \frac{1}{z^2} + \frac{5i}{z-2} + \frac{1}{z-1-2i}$ is analytic on the punctured disk $0 < |z| < 0.3$ (inside the circle C_0). Since $\frac{5i}{z-2} + \frac{1}{z-1-2i}$ is also analytic at zero, the Laurent series of $f(z)$ has the form

$$f(z) = \frac{3}{z} + \frac{1}{z^2} + \sum_{n=0}^{\infty} a_n z^n$$

We conclude that $f(z)$ has a pole of order 2 at $z_0 = 0$ and residue $\operatorname*{Res}_{z=0} f(z) = 3$. Similarly, $f(z)$ has simple poles (order 1) at $z_1 = 2$ and $z_2 = 1 + 2i$ with

$$\operatorname*{Res}_{z=2} f(z) = 5i, \qquad \operatorname*{Res}_{z=1+2i} f(z) = 1$$

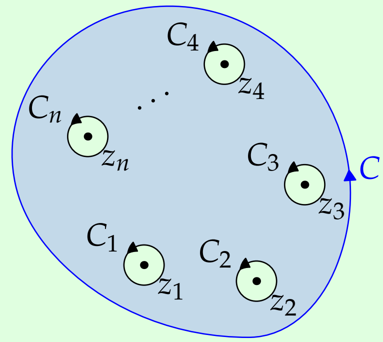
3. $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$ has an essential singularity at zero with $\operatorname*{Res}_{z=0} e^{1/z} = 1$.

Theorem 6.4 (Cauchy's Residue Theorem). *If $f(z)$ is analytic on and inside a simple closed C, except at finitely many singular points $z_1, \ldots, z_n$, then*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

More generally, if C is closed and orbits the point z_k counter-clockwise w_k times (the winding number*), then*

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} w_k \operatorname*{Res}_{z=z_k} f(z)$$



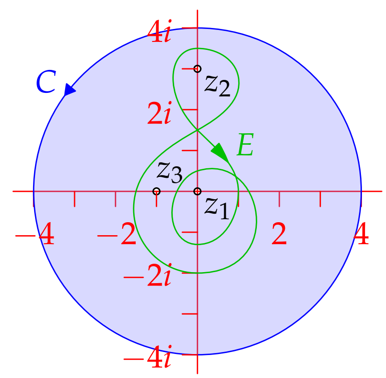
Proof. (Simple case). Center a small circle C_k at each z_k such that no other singularities lie on or inside C_k. Now apply Cauchy–Goursat to the domain on and between C and $C_1 \cup \cdots \cup C_n$. ∎

Examples 6.5. Let C be the circle with radius 4 centered at the origin and E the green curve drawn.

1. The function $f(z) = \frac{3(1+iz)}{z(z-3i)}$ has simple poles at $z_1 = 0$ and $z_2 = 3i$. There are several ways to compute the residues and thus the integrals $\oint_C f(z)\mathrm{d}z$ and $\oint_E f(z)\mathrm{d}z$.

   Partial Fractions For *this* example, this is very easy.

$$f(z) = \frac{i}{z} + \frac{2i}{z-3i} \implies \operatorname*{Res}_{z=0} f(z) = i, \quad \operatorname*{Res}_{z=3i} f(z) = 2i$$
$$\implies \oint_C f(z)\,\mathrm{d}z = 2\pi i(i+2i) = -6\pi$$



   The curve E is the union of three closed curves; twice *clockwise* around z_1 and once *counter-clockwise* around z_2. Therefore

$$\int_E f(z)\,\mathrm{d}z = 2\pi i \left[-2 \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=3i} f(z)\right] = 0$$

   Laurent series Remember that we only need the z^{-1} terms for the residues!

$$\frac{3(1+iz)}{z(z-3i)} = \frac{i-z}{z(1-\frac{iz}{3})} = \left(iz^{-1}-1\right)\sum_{n=0}^{\infty}\left(\frac{iz}{3}\right)^n = \frac{i}{z} + \text{power series}$$
$$\frac{3(1+iz)}{z(z-3i)} = \frac{z-3i+2i}{(1+\frac{z-3i}{3i})(z-3i)} = \left(\frac{2i}{z-3i}+1\right)\sum_{n=0}^{\infty}\left(\frac{3i-z}{3i}\right)^n = \frac{2i}{z-3i} + \text{power series}$$

   Cauchy's formula Let C_k be a small circle around z_k, then

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_1} \frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z-3i}\right|_{z=0} = i$$
$$\operatorname*{Res}_{z=3i} f(z) = \frac{1}{2\pi i}\oint_{C_2} f(z)\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{C_2} \frac{3(1+iz)}{z(z-3i)}\,\mathrm{d}z = \left.\frac{3(1+iz)}{z}\right|_{z=3i} = 2i$$

   We'll revisit this last approach in the next section.

2. Plainly $f(z) = z^2 \sin \frac{1}{z}$ has one isolated singularity at the origin. Using the Maclaurin series for $\sin z$, we see that this is an essential singularity, and can easily evaluate the required integrals:

$$z^2 \sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n} \implies \oint_C z^2 \sin \frac{1}{z} \, dz = 2\pi i \operatorname*{Res}_{z=0} \left(z^2 \sin \frac{1}{z} \right) = -\frac{\pi i}{3}$$

Since E loops twice *clockwise* around the origin, we obtain

$$\int_E z^2 \sin \frac{1}{z} \, dz = 2\pi i \cdot (-2) \operatorname*{Res}_{z=0} \left(z^2 \sin \frac{1}{z} \right) = \frac{2\pi i}{3}$$

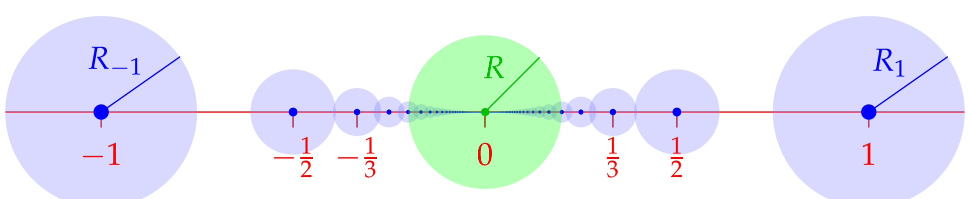
3. The function $f(z) = 3e^{1/z} + \frac{4}{z-7i} + \frac{2i}{z+1}$ has an essential singularity at the origin and simple poles at -1 and $7i$. Since the last of these lies *outside* the curves C, E, it does not contribute to either integral. Moreover, note that E loops *twice* clockwise around the origin and *once* clockwise around $z_3 = -1$. We therefore have

$$\oint_C f(z) \, dz = 2\pi i \left(\operatorname*{Res}_{z=0} 3e^{1/z} + \operatorname*{Res}_{z=-1} \frac{2i}{z+1} \right) = 2\pi i (3 + 2i)$$
$$\oint_E f(z) \, dz = 2\pi i \left(-2 \operatorname*{Res}_{z=0} 3e^{1/z} - \operatorname*{Res}_{z=-1} \frac{2i}{z+1} \right) = 2\pi i (-6 - 2i) = 4\pi (1 - 3i)$$

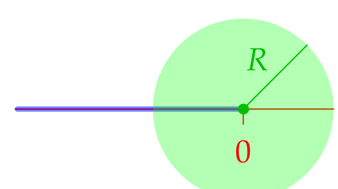
Non-isolated Singularities

Consider the converse of Definition 6.2: a singularity z_0 is non-isolated if *every* punctured disk $0 < |z - z_0| < R$ contains at least one point where $f(z)$ is non-analytic. Necessarily this requires $f(z)$ to be non-analytic at infinitely many points. Non-isolated singularities typically appear in two flavors.

Examples 6.6. 1. The function $f(z) = \left(e^{\frac{2\pi i}{z}} - 1\right)^{-1}$ has singularities at $z_0 = 0$ and whenever $z_n = \frac{1}{n}$ for each non-zero integer n. Each of the non-zero singularities is isolated (e.g. choose $R_n = \frac{1}{(|n|+1)^2}$). The remaining *limit point* $z_0 = 0$ is a non-isolated singularity: for any $R > 0$, the punctured disk $0 < |z| < R$ contains non-zero singularities.



2. The function $f(z) = \sqrt{z}$ has *branch point* $z_0 = 0$. For f to be analytic, we need to make a branch cut: for instance, the principal value of $\sqrt{z}$ has the non-positive real axis as a branch cut.
Since any domain $0 < |z| < R$ contains other points of the branch cut (where f is non-analytic), it follows that a branch point is a non-isolated singularity *for any branch* of f.



Exercises 6.1 1. For each of the following types of singularity, what, if anything, can you say about the value of the residue $\operatorname*{Res}_{z=z_0} f(z)$? Choose from 'Equals zero,' 'Non-zero,' 'No restriction'.

(a) Removable singularity.

(b) Simple pole.

(c) Pole of order $m \geq 2$.

(d) Essential singularity.

2. Find the residue at $z = 0$ of each function:

(a) $\dfrac{1}{z+3z^2}$ (b) $z\cos\dfrac{1}{z}$ (c) $\dfrac{z-\sin z}{z}$

3. Let C be the circle $|z| = 3$. Evaluate the integrals using Cauchy's residue theorem:

(a) $\oint_C \dfrac{e^{-z}}{z^2}\,dz$ (b) $\oint_C \dfrac{e^{-z}}{(z-1)^2}\,dz$ (c) $\oint_C z^2 e^{1/z}\,dz$ (d) $\oint_C \dfrac{z+1}{z^2-2z}\,dz$

4. Suppose a closed contour C loops twice counter-clockwise around $z = i$ and three times clockwise around $z = 2$. Use residues to compute the integral

$$\int_C \frac{z+3}{(z-2)^2(z-i)}\,dz$$

5. Identify the type of singular point of each of the following functions and determine the residue:

(a) $\dfrac{1-\cosh z}{z^3}$ (b) $\dfrac{1-e^{2z}}{z^4}$ (c) $\dfrac{e^{2z}}{(z-1)^2}$

6. Suppose $f(z)$ is analytic at z_0 and define $g(z) = (z-z_0)^{-1}f(z)$. Prove:

(a) If $f(z_0) \neq 0$, then z_0 is a simple pole of $g(z)$ with $\operatorname*{Res}_{z=z_0} g(z) = f(z_0)$;

(b) If $f(z_0) = 0$, then z_0 is a removable singularity of $g(z)$.

7. Let $P(z)$ and $Q(z)$ be polynomials whose degrees satisfy $2 + \deg P \leq \deg Q$ and assume C is a simple closed contour such that all zeros of $Q(z)$ lie interior to C.

(a) Prove that $\oint_C \dfrac{P(z)}{Q(z)}\,dz = 0$

(*Hint: Try the substitution* $w = \frac{1}{z}$)

(b) What can you conclude if $\deg Q = \deg P + 1$?

6.2 Poles & Zeros

Recall Example 6.5 where we used Cauchy's integral formula as one of the methods for computing a residue. If the order of a pole is known, this approach is often fairly efficient.

Theorem 6.7. *A function $f(z)$ has a pole of order m at z_0 if and only if $f(z) = (z - z_0)^{-m}\phi(z)$ where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. In such a case,* [a]

$$f(z) = (z - z_0)^{-m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z - z_0)^n \implies \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\phi^{(m-1)}(z_0)$$

This specializes to $\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$ for a simple pole.

[a] This formula works even if $\phi(z_0) = 0$; you only need $\phi(z)$ analytic and *non-constant* at z_0: a naïve application means you won't know the order of the pole and you'll have to differentiate more times than necessary!

Examples 6.8. 1. (Example 6.5) Write $f(z) = \frac{3(1+iz)}{z(z-3i)} = \frac{\phi_1(z)}{z} = \frac{\phi_2(z)}{z-3i}$ and verify:

Simple pole at $z_1 = 0$: $\phi_1(z) = \frac{3(1+iz)}{z-3i}$ is analytic, and $\phi_1(0) = \frac{3}{-3i} = i = \operatorname*{Res}_{z=0} f(z)$

Simple pole at $z_2 = 3i$: $\phi_2(z) = \frac{3(1+iz)}{z}$ is analytic, and $\phi_2(3i) = \frac{3(1-3)}{3i} = 2i = \operatorname*{Res}_{z=3i} f(z)$

2. Write $f(z) = \frac{1-2iz}{(z-1)(z-2i)^3} = \frac{\phi_1(z)}{z-1} = \frac{\phi_2(z)}{(z-2i)^3}$ and compute:

Simple pole at $z_1 = 1$: $\phi_1(z) = \frac{1-2iz}{(z-2i)^3}$ is analytic and non-zero at $z_1 = 1$. It follows that

$$\operatorname*{Res}_{z=1} f(z) = \phi_1(1) = \frac{1-2i}{(1-2i)^3} = \frac{1}{(1-2i)^2} = \frac{4i-3}{25}$$

Pole of order three at $z_2 = 2i$: $\phi_2(z) = \frac{1-2iz}{z-1} = -2i + \frac{1-2i}{z-1}$ is analytic and non-zero at $2i$, and

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(3-1)!}\phi_2''(2i) = \frac{1-2i}{(z-1)^3}\bigg|_{z=2i} = \frac{-1}{(2i-1)^2} = \frac{3-4i}{25}$$

Proof. ($\Rightarrow$) By Laurent's Theorem, $f(z)$ equals its Laurent series. It moreover has a pole of order m at z_0 if and only if

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = (z-z_0)^{-m}\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \text{ where } a_{-m} \neq 0 \qquad (*)$$

Plainly $\phi(z) := \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$ is analytic at z_0 and satisfies $\phi(z_0) = a_{m-n} \neq 0$.

($\Leftarrow$) Taylor's Theorem says that $\phi(z)$ equals its Taylor series $\sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$, whence (uniqueness of representation) $(*)$ is *the* Laurent series of $f(z)$ and $f(z)$ has a pole of order m. ∎

As the examples show, the method is very effective when $f(z)$ is rational with low-order poles; as a bonus, it saves us from partial fractions! Its utility is more variable for other functions...

Examples 6.9. 1. Taking $\phi(z) = \frac{e^z}{z+1}$ shows that the non-rational function

$$f(z) = \frac{e^z}{(z-1)^2(z+1)} = \frac{\phi(z)}{(z-1)^2}$$

has a pole of order two at $z_0 = 1$ and moreover that

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\phi'(1) = \left.\frac{(z+1)e^z - e^z}{(z+1)^2}\right|_{z=1} = \frac{1}{4}e$$

2. Don't let the denominator fool you! At first glance we appear to have a pole of order *six*:

$$f(z) = \frac{6\sin z - 6z + z^3}{z^6} = \frac{\widetilde{\phi}(z)}{z^6} \overset{??}{\Longrightarrow} \operatorname*{Res}_{z=0} f(z) = \frac{1}{5!}\widetilde{\phi}^{(5)}(0) = \frac{6}{120} = \frac{1}{20}$$

However, if we apply the Maclaurin series for sine, we instead find a *simple pole*:

$$\begin{aligned} f(z) &= \frac{1}{z^6}\left(6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} - 6z + z^3\right) = z^{-6}\sum_{n=2}^{\infty}\frac{6(-1)^n}{(2n+1)!}z^{2n+1} \\ &= z^{-1}\sum_{m=0}^{\infty}\frac{6(-1)^m}{(2m+5)!}z^{2m} = \frac{1}{20z} + \frac{1}{840} + \cdots \Longrightarrow \operatorname*{Res}_{z=0} f(z) = \frac{1}{20} \end{aligned}$$

Even though the residue was correct, our original $\widetilde{\phi}$ was wrong ($\widetilde{\phi}(0) \neq 0$). The correct function is the series $\phi(z) = 6\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+5)!}z^{2m}$.

Zeros of Analytic Functions

It turns out that poles and zeros of analytic functions are intimately related. We start by mirroring Definition 6.2 and Theorem 6.7.

Definition 6.10. Suppose $f(z)$ is analytic at z_0 and $f(z_0) = 0$. We say that z_0 is a *zero of order* m if $f^{(m)}(z_0)$ is the first *non-zero* derivative. We refer to a *simple zero* when $m = 1$.

A zero z_0 is *isolated* if it has some neighborhood with no other zeros:

$$\exists R > 0 \text{ such that } 0 < |z - z_0| < R \implies f(z) \neq 0$$

We are used to the idea of polynomial having a zero z_0 if and only if we can factorize out $z - z_0$. The tight link-up with Taylor series makes essentially this observation hold for *any* analytic function!

Lemma 6.11. *A function $f(z)$ has a zero z_0 of order m if and only if $f(z) = (z-z_0)^m\psi(z)$ where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Indeed, on some disk $|z - z_0| < R_0$,*

$$f(z) = \sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = (z-z_0)^m\psi(z)$$

Examples 6.12. 1. $f(z) = z^4(z-2i)^{10} = z^4\psi_1(z) = (z-2i)^{10}\psi_2(z)$ has two zeros:

Order four at $z_1 = 0$: $\psi_1(z) = (z-2i)^{10} \implies \psi_1(0) = -1024 \neq 0$.

Order ten at $z_2 = 2i$: $\psi_2(z) = z^4 \implies \psi_2(2i) = 16 \neq 0$.

2. $g(z) = 17(z-4i)^3 \cos z$ has a zero of order three at $4i$, and simple zeros at each half-integer multiple $\left(\frac{1}{2} \pm k\right)\pi$ of π. For instance

$$\begin{aligned} g(z) &= 17(z-4i)^3 \cos\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right) = -17(z-4i)^3 \sin\left(z - \frac{\pi}{2}\right) \\ &= -17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{2n+1} \\ &= \left(z - \frac{\pi}{2}\right)\left[-17(z-4i)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(z - \frac{\pi}{2}\right)^{2n}\right] = \left(z - \frac{\pi}{2}\right)\psi(z) \end{aligned}$$

The examples are typical: just as with singularities, the typical arrangement is for a zero of an analytic function to be *isolated.* Indeed, analytic functions with non-isolated zeros are very boring...

Theorem 6.13. *Let z_0 be a zero of an analytic function $f(z)$. The following are equivalent:*

1. *$f(z)$ is non-zero at some point of every neighborhood $|z - z_0| < \epsilon$ of z_0.*
2. *z_0 is a zero of some positive order m.*
3. *z_0 is an isolated zero.*

The distinction between conditions 1 and 3 is important: the first is weaker and the the equivalence is *false* for non-analytic functions. For example, at $z_0 = 0$ the non-analytic function $f(z) = z + \bar{z} = 2x$ satisfies condition 1 but not 3 (e.g. $f(\frac{\epsilon}{2}) \neq 0$). There is something to prove here!

Proof. ($1 \Rightarrow 2$) The Taylor series of $f(z)$ is non-zero, else $f(z)$ would be zero on some disk $|z - z_0| < \epsilon$. There must therefore be some minimum $m \in \mathbb{N}$ such that $f^{(m)}(z_0) \neq 0$, whence z_0 is a zero of order m.

($2 \Rightarrow 3$) $f(z) = (z - z_0)^m \psi(z)$ where $\psi(z)$ is analytic and $\psi(z_0) \neq 0$. Since $\psi(z)$ is continuous, it is non-zero on some disk $|z - z_0| < \epsilon$, and so also is $f(z)$. We conclude that z_0 is an isolated zero.

($3 \Rightarrow 1$) This is trivial.

∎

Corollary 6.14. *If $f(z)$ is analytic on a connected open domain D containing z_0, and $f(z) = 0$ at each point of some contour C containing z_0, then $f(z) \equiv 0$ on D.*

Proof. This is the negation of the situation in the Theorem: plainly z_0 is not isolated and so $f(z) \equiv 0$ on some disk centered on z_0. The usual patching argument extends this to D.

∎

This essentially proves the result regarding unique analytic continuations from earlier in the course.

Reciprocals switch poles and zeros

It seems intuitive that we can turn poles into zeros just by flipping a function upside down.

Theorem 6.15. *Let $f(z)$ be analytic at z_0 and $g(z) = \frac{1}{f(z)}$. Then, at z_0,*

$$f(z) \text{ has a zero of order } m \iff g(z) \text{ has a pole of order } m$$

The proof is an easy exercise in combining Theorem 6.7 and Lemma 6.11.

This approach yields a very quick method for computing residues of functions with *simple poles*, and is particularly useful which a function is multi-valued.

Corollary 6.16. *Suppose p, q are analytic at z_0 such that $f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z-z_0)\psi(z)}$ has a simple pole at z_0. Plainly $q'(z_0) = \psi(z_0)$, from which*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Examples 6.17.

1. The function $f(z) = \frac{p(z)}{q(z)} = \frac{(z^2+1)^2}{z-3} = \frac{(z-i)^2(z+i)^2}{z-3}$ has zeros of order two at $\pm i$ and a simple pole at $z = 3$. The reciprocal has the reverse arrangement: poles of order two at $\pm i$ and a simple zero at 3. Moreover,

$$\operatorname*{Res}_{z=3} f(z) = \frac{p(3)}{q'(3)} = \frac{(3^2+1)^2}{1} = 100$$

2. The function $g(z) = \frac{p(z)}{q(z)} = \frac{\sin z}{z^2+4} = \frac{\sin z}{(z-2i)(z+2i)}$ has simple poles at $\pm 2i$ with

$$\operatorname*{Res}_{z=2i} f(z) = \operatorname*{Res}_{z=-2i} f(z) = \frac{p(\pm 2)}{q'(\pm 2)} = \frac{\sin 2i}{4i} = \frac{1}{8}(e^2 - e^{-2}) = \frac{1}{4}\sinh 2$$

The reciprocal has simple poles at $z = n\pi$ for every $n \in \mathbb{Z}$: moreover

$$\operatorname*{Res}_{z=n\pi} \frac{1}{f(z)} = \frac{q(n\pi)}{p'(n\pi)} = \frac{n^2\pi^2+4}{\cos n\pi} = (-1)^n(n^2\pi^2+4)$$

3. Since $q(z) = e^{2z} - 1 = \sum_{n=1}^{\infty} \frac{2^n z^n}{n!} = z\sum_{n=0}^{\infty} \frac{2^n z^n}{(n+1)!}$ has a simple zero at $z = 0$, we see that

$$f(z) = \frac{\sqrt{z+4i}}{(z+i)^2 \operatorname{Log}(z+2)(e^{2z}-1)} = \frac{p(z)}{q(z)}$$

has a simple pole at $z = 0$ (we use the principal value of $\sqrt{z+4i}$). Moreover

$$\operatorname*{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{e^{\frac{i\pi}{4}}}{\ln 2} = \frac{1+i}{\sqrt{2}\ln 2}$$

We could instead have chosen $q(z) = (z+i)^2 \operatorname{Log}(z+2)(e^{2z}-1)$, but the differentiation would have been much worse!

Counting Poles and Zeros

Definition 6.18. A function f is *meromorphic* on a domain D if it is analytic except at isolated *poles*. That is, f cannot have essential (or removable) singularities.

Theorem 6.19. *Suppose C is a positively oriented closed contour. If f is meromorphic on and inside C, then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = Z - P$$

where Z and P are the number of zeros and poles of f inside C, counted up to multiplicity.

Example 6.20. Consider, $f(z) = \frac{(z-i)^2 \sin z}{(z-5)^4}$ where C is a large circle surrounding the points 0, i and 5. Plainly $Z = 2+1 = 3$ and $P = 4$. By an unpleasant application of the quotient rule (or better using logarithms, just be careful!), we obtain

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_C \frac{2}{z-i} + \frac{\cos z}{\sin z} - \frac{4}{z-5}\,\mathrm{d}z = 2 + \cos 0 - 4 = -1 = Z - P$$

Proof. If f has a zero of order m at $z = z_k$, then $f(z) = (z-z_k)^m \psi(z)$ on and inside a some small circle C_k centered at z_k, where $\psi(z)$ is analytic and *non-zero*. But then

$$\begin{aligned}\oint_{C_k} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \oint_{C_k} \frac{m(z-z_0)^{m-1}\psi(z) + (z-z_0)^m \psi'(z)}{(z-z_0)^m \psi(z)}\,\mathrm{d}z = \oint_{C_k} \frac{m}{z-z_0} + \frac{\psi'(z)}{\psi(z)}\,\mathrm{d}z \\ &= 2\pi i m\end{aligned}$$

Similarly, if f has a pole of order m, then we repeat with $f(z) = (z-z_k)^{-m}\phi(z)$ to obtain

$$\begin{aligned}\oint_{C_k} \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \oint_{C_k} \frac{-m(z-z_0)^{-m-1}\phi(z) + (z-z_0)^{-m} \phi'(z)}{(z-z_0)^{-m} \phi(z)}\,\mathrm{d}z = \oint_{C_k} -\frac{m}{z-z_0} + \frac{\phi'(z)}{\phi(z)}\,\mathrm{d}z \\ &= -2\pi i m\end{aligned}$$

Cauchy's residue theorem completes the proof. ∎

Properties of Singularities

We finish by considering some equivalent conditions for the various types of singularities.

Theorem 6.21 (Removable Singularities). *Suppose $f(z)$ has an isolated singularity at z_0 (and is therefore analytic on a punctured disk $0 < |z| < R$). The following are equivalent:*

1. *The singularity is removable.*
2. $\lim_{z \to z_0} f(z)$ *exists and is finite.*
3. $f(z)$ *is bounded on some* $0 < |z - z_0| < \delta$.

Proof. For simplicity, suppose that $z_0 = 0$.

(1 ⇒ 2) $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $0 < |z| < R$, whence $\lim_{z\to 0} f(z) = a_0$.

(2 ⇒ 3) This is almost tautological but it bears repeating: If $\lim_{z\to 0} f(z) = a_0$ exists and is finite then choose $\epsilon = |a_0|$ in the definition;

$$\begin{aligned}\exists\delta > 0 \text{ such that } 0 < |z| < \delta &\implies |f(z) - a_0| < |a_0| \\ &\implies |f(z)| = |f(z) - a_0 + a_0| \le 2\,|a_0|\end{aligned}$$

(3 ⇒ 1) Consider

$$g(z) = \begin{cases} z^2 f(z) & \text{if } 0 < z < \delta \\ 0 & \text{if } z = 0 \end{cases}$$

Since f is bounded, we may compute the limit

$$\lim_{z\to 0} \frac{g(z) - g(0)}{z} = \lim_{z\to 0} z f(z) = 0$$

whence $g(z)$ is differentiable at zero! Since it is already differentiable on the punctured neighborhood $0 < |z| < \min\{\delta, R\}$, it is therefore analytic on the *disk* $|z| < \min\{\delta, R\}$ and equals its Maclaurin series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

However $g(0) = 0 = g'(0) \implies b_0 = b_1 = 0$ and so

$$g(z) = z^2 \sum_{n=0}^{\infty} b_{n-2} z^n \implies f(z) = \sum_{n=0}^{\infty} b_{n-2} z^n \text{ on } 0 < |z| < \min\{\delta, R\}$$

whence f has a removable singularity. ∎

We leave the remaining results as exercises.

Theorem 6.22. *Suppose f has an isolated singularity at $z = z_0$.*

1. *z_0 is a pole if and only if $\lim_{z\to z_0} f(z) = \infty$.*
2. *If z_0 is essential and $w \in \mathbb{C} \cup \{\infty\}$, then $\exists z_n \to z_0$ such that $f(z_n) \to w$.*

This second result is the *Casorati–Weierstrass Theorem*; the range of $f(z)$ is *dense* in a neighborhood of an essential singularity. A stronger result is available, through its proof is beyond us.

Theorem 6.23 (Picard). *If z_0 is an essential singularity of $f(z)$, then $f(z)$ takes every complex value* except at most one *in any neighborhood of z_0.*

Example 6.24. Let $f(z) = e^{1/z}$ at $z_0 = 0$. If $w = e^{1/z}$, write $w = re^{i\theta}$ with $0 \leq \theta < 2\pi$, from which

$$e^{\ln r + i\theta} = e^{1/z} \implies \frac{1}{z} = \ln r + i\theta + 2\pi i n$$

for any integer n. If $n > 0$, observe that

$$\left|\frac{1}{z}\right| = \sqrt{(\ln r)^2 + (\theta + 2\pi n)^2} > 2\pi n$$

whence $|z|$ can be chosen arbitrarily small. A suitable z thus exists in any punctured disk $0 < |z| < \delta$.

Exercises 6.2

1. Determine the order of each pole and its residue.

   (a) $f(z) = \dfrac{z+1}{z^2+9}$ (b) $f(z) = \left(\dfrac{z}{2z+1}\right)^3$

2. Show that:

   (a) $\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ when $|z| > 0$ and $\arg z \in (0, 2\pi)$

   (b) $\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$

   (c) $\operatorname*{Res}_{z=z_n} \big(z \sec z\big) = (-1)^{n+1} z_n$, where $z_n = \frac{\pi}{2} + n\pi$ and $n \in \mathbb{Z}$

3. Find the value of the integral

   $$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

   when C is each of the circles: (a) $|z-2| = 2$ and (b) $|z| = 4$.

4. Let C be the circle $|z| = 2$. Evaluate $\oint_C \tan z\, dz$.

5. Prove Theorem 6.15.

6. Let $f(z) = \left(z \sin \frac{\pi}{z}\right)^{-1}$

   (a) Evaluate $\operatorname*{Res}_{z=\frac{1}{n}} f(z)$ for each $n \in \mathbb{Z}$.

   (b) Why doesn't $\operatorname*{Res}_{z=0} f(z)$ make sense?

7. Suppose $f(z)$ is analytic and non-constant at z_0. Prove that

   $$\exists \epsilon > 0 \text{ such that } 0 < |z - z_0| < \epsilon \implies f(z) \neq f(z_0)$$

8. Suppose that C is the rectangle whose sides are the lines $x = \pm 2, y = 0$ and $y = 1$. Prove that

   $$\oint_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$$

   (*Hint: the integrand has four simple poles, only two of which lie inside* C)

The last two questions are more of a challenge

9. Prove Theorem 6.22. For simplicity, assume $z_0 = 0$.

   Hint 1: $f(z)$ *has a pole if and only if* $\frac{1}{f(z)}$ *has a zero.*

   Hint 2: If no such sequence exists, show that $g(z) := \frac{1}{f(z)-w}$ *is analytic and bounded.*

10. Suppose $f(z)$ is analytic on and inside a simple closed curve C, and that it has no zeros on C. We consider the integral

$$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

   (a) Explain why I counts the number of zeros (including multiplicity) of $f(z)$ inside C.

   (b) (Winding number) Explain why I also counts the number times the curve $\gamma = f(C)$ orbits the origin counter-clockwise.
   (*Hint: let* $w = f(z)$)

   (c) (Rouche's Theorem) Suppose that $|g(z)| < |f(z)|$ for all $z \in C$. Prove that the number of zeros of $f + g$ inside C equals that of f.
   (*Hint: Apply part (a) to the product* $f + g = f \cdot (1 + \frac{g}{f})$ *and consider why the function* $1 + \frac{g}{f}$ *has winding number zero*)

   (d) Since $\left|4z^3\right| > \left|z^{22} + 2i\right|$ on the circle $|z| = 1$, how many solutions (up to multiplicity) are there to the equation $z^{22} + 4z^3 + 2i = 0$ on the domain $|z| < 1$?

6.3 Improper Integrals

We describe a natural application of residues to the evaluation of certain *real* improper integrals. We start with an alternative definition of improper integral more suited to our purposes.

Definition 6.25. Provided the limit exists, the *Cauchy principal value* of $\int_{-\infty}^{\infty} f(x)\,dx$ is the limit

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

This is a potentially misleading interpretation of the improper integral. In standard calculus the definition requires *two* limits, *both* of which must exist for the integral to converge:

$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\,dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)\,dx$$

If $\int_{-\infty}^{\infty} f(x)\,dx$ converges, then it certainly equals its Cauchy principal value. However, the converse isn't true.

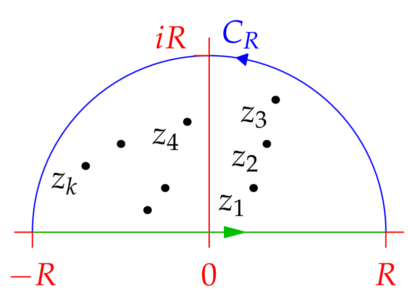
Example 6.26. If $f(x)$ is *any* odd function $\big(f(-x) = -f(x)\big)$, then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty} 0 = 0$$

If either of the 1-sided improper integrals diverges, then the the full integral also diverges: e.g.

$$\text{P.V.} \int_{-\infty}^{\infty} x^3\,dx = 0 \quad \text{but} \quad \int_{0}^{\infty} x^3\,dx \text{ diverges} \implies \int_{-\infty}^{\infty} x^3\,dx \text{ diverges}$$

Residue theory supplies a neat trick for computing Cauchy principal values:



1. Suppose $f(x)$ is the restriction to the real line of a *complex function* $f(z)$ which is analytic except at finitely many poles $z_1, \ldots, z_n$ in the upper half-plane $\operatorname{Im} z > 0$;
2. Choose $R > 0$ so that $R > |z_k|$ for each k and let C_R be the counter-clockwise upper semi-circle centered at the origin with radius R. By Cauchy's Residue Theorem,

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

3. If $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$, then $\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$

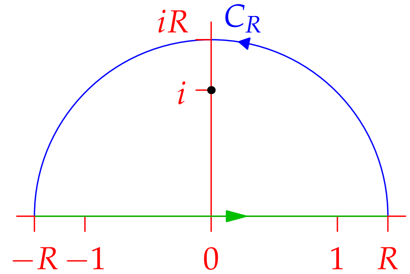
Mostly we will apply the method to rational functions $f(z) = \frac{p(z)}{q(z)}$, though it works more generally. Beyond ease of residue calculation, the reason is that $\deg q \geq \deg p + 2$ is enough to guarantee that step 3 applies (Exercise 5).

Examples 6.27. 1. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$ has simple poles at $\pm i$. Provided $|z| = R > 1$,

$$\left|z^2+1\right| \geq \left||z|^2-1\right| = R^2-1 \implies \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$$

$$\implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{2\pi i}{2i} = \pi$$



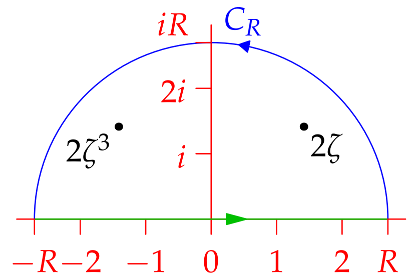
Compare with the usual calculus method:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1} x\Big|_{-R_1\to-\infty}^{R_2\to\infty} = \pi$$

2. $f(z) = \frac{4(z^2-1)}{z^4+16}$ has simple poles at $\pm 2\zeta, \pm 2\zeta^3$ where $\zeta = e^{\frac{\pi i}{4}}$. Let $p(z) = 16(z^2-1)$ and $q(z) = z^4+16$, so that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^2-1}{z_0^3}$$

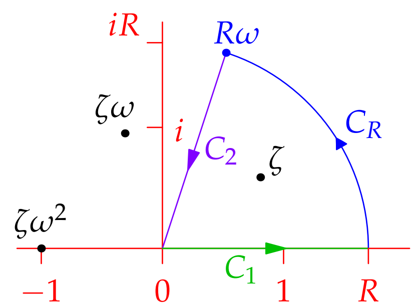
When $|z| = R > 2$, we see that



$$\left|z^4+16\right| \geq R^4-16 \implies \left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \leq \frac{4\pi R(R^2+1)}{R^4-16} \to 0$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \left(\operatorname*{Res}_{z=2\zeta} f(z) + \operatorname*{Res}_{z=2\zeta^3} f(z)\right) = 2\pi i \left(\frac{4\zeta^2-1}{8\zeta^3} + \frac{4\zeta^6-1}{8\zeta^9}\right) = \frac{3\pi}{2\sqrt{2}}$$

3. Variations are possible, for instance by taking only part of a semi-circular arc. The function $f(z) = \frac{1}{z^5+1}$ has five simple poles: the fifth-roots of -1.
Since the pole $\zeta\omega^2 = -1$ lies on the negative real axis, the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ diverges. Instead consider the arcs in the picture when $R > 1$. Parametrizing C_2 via $z(t) = t\omega$,



$$\int_{C_2} \frac{1}{z^5+1}\,\mathrm{d}z = \int_R^0 \frac{\omega}{t^5+1}\,\mathrm{d}t = -\omega \int_0^R \frac{1}{t^5+1}\,\mathrm{d}t$$

$$= -\omega \int_{C_1} \frac{1}{z^5+1}\,\mathrm{d}z$$

$$\implies (1-\omega) \int_0^R \frac{1}{x^5+1}\,\mathrm{d}x + \int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=\zeta} \frac{1}{z^5+1} = \frac{2\pi i}{5\zeta^4} = \frac{2\pi i}{5\omega^2}$$

When $|z| = R > 1$, we see that $\left|z^5+1\right| \geq R^5-1 \implies \left|\int_{C_R} \frac{1}{z^5+1}\,\mathrm{d}z\right| \leq \frac{2\pi R}{5(R^5-1)} \xrightarrow[R\to\infty]{} 0$, and we conclude

$$\int_0^{\infty} \frac{1}{x^5+1}\,\mathrm{d}x = \frac{2\pi i}{5(\omega^2-\omega^3)} = \frac{2\pi i}{5\zeta\omega^2(\zeta^{-1}-\zeta)} = \frac{2\pi i}{5(2i\sin\frac{\pi}{5})} = \frac{\pi}{5}\csc\frac{\pi}{5}$$

Jordan's Lemma

It is often useful, particularly when computing Fourier transforms,[1] to evaluate integrals of the form

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = \int_{-\infty}^{\infty} f(x)\cos ax\,dx + i\int_{-\infty}^{\infty} f(x)\sin ax\,dx$$

where $a > 0$ is a real constant and $f : \mathbb{R} \to \mathbb{C}$ is a given function. If $f(x)$ is real-valued, then the above breaks the integral into real and imaginary parts. Given reasonable conditions on $f(x)$, the above method can often be employed.

Example 6.28. The function $f(z) = \frac{e^{3iz}}{z^2+4}$ is analytic on the upper half-plane except at the simple pole $z = 2i$. With $R > 2$ and C_R the usual semi-circle, we see that

$$\left|e^{3iz}\right| = e^{-3y} \le 1 \implies \left|\int_{C_R} f(z)e^{3iz}\,dz\right| \le \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{e^{3ix}}{x^2+4}\,dx = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{3iz}}{z^2+4} = 2\pi i\frac{e^{-6}}{4i} = \frac{1}{2}\pi e^{-6}$$

Since this is real, we see that the result is in fact the integral $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx$. We don't need the Cauchy principal value of the integral here since the full improper integral converges. The corresponding imaginary integral is trivially zero since $\frac{\sin x}{x^2+1}$ is an odd function.

To assist with these computations, we state the following result without proof.

Theorem 6.29 (Jordan's Lemma). *Let $a, R_0 > 0$ be given and suppose $f(z)$ is analytic at all points exterior to C_{R_0} in the upper half-plane. Suppose also that*

$$\forall R > R_0,\ \exists M_R \text{ such that } |f(z)| \le M_R \text{ and } \lim_{R\to\infty} M_R = 0$$

Then $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}\,dz = 0$. If $f(z)$ also satisfies the hypotheses of our method, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i\sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)e^{iaz}$$

Example 6.30. If $f(x) = \frac{x+1}{x^2+9}$ and $R > 3$, then

$$|f(z)| = \frac{|z+2|}{|z^2+9|} \le \frac{R+2}{R^2-9} = M_R \xrightarrow[R\to\infty]{} 0$$

$$\implies \text{P.V.}\int_{-\infty}^{\infty} \frac{(x+2)e^{iax}}{x^2+9}\,dx = 2\pi i \operatorname*{Res}_{z=3i} \frac{(z+2)e^{iaz}}{z^2+9} = \frac{2\pi i(2+3i)e^{-3a}}{6i} = \frac{\pi(2+3i)}{3}e^{-3a}$$

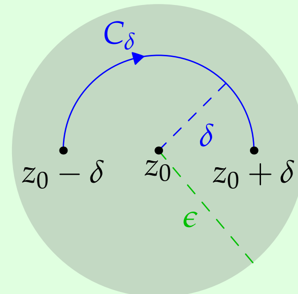
By considering even and odd functions, etc., we can rewrite this as

$$\int_0^{\infty} \frac{\cos ax}{x^2+9}\,dx = \frac{\pi}{6}e^{-3a} \qquad \int_0^{\infty} \frac{x\sin ax}{x^2+9}\,dx = \frac{\pi}{2}e^{-3a}$$

[1] The Fourier transform of $f(x)$ is the function $\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx$.

Indented paths Another modification allows $f(z)$ to have a simple pole on the real axis.

Lemma 6.31. *Let D be the disk $|z - z_0| \le \epsilon$, let $\delta < \epsilon$, and let C_δ be the clockwise semi-circle in the picture.*



1. *If $\phi(z)$ is analytic on D, then $\lim_{\delta \to 0} \int_{C_\delta} \phi(z)\,dz = 0$.*
2. *If $f(z)$ is analytic on $D \setminus \{z_0\}$ with a simple pole at z_0, then*

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)\,dz = -\pi i \operatorname*{Res}_{z=z_0} f(z)$$

More generally, if C_δ spans θ radians clockwise round z_0, then $\lim_{\delta \to 0} \int_{C_\delta} f(z)\,dz = -i\theta \operatorname*{Res}_{z=z_0} f(z)$.

Proof. 1. ϕ is continuous on D and thus bounded by some M: but then

$$\left| \int_{C_\delta} \phi(z)\,dz \right| \le M\pi\delta$$

2. The Laurent series expansion of $f(z)$ on $D \setminus \{z_0\}$ is

$$f(z) = \frac{a_{-1}}{z - z_0} + \phi(z)$$

where $a_{-1} = \operatorname*{Res}_{z=z_0} f(z)$ and $\phi(z)$ is analytic on D. Now evaluate

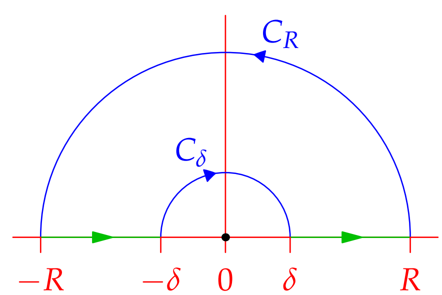
$$\int_{C_\delta} \frac{a_{-1}}{z - z_0}\,dz = a_{-1} \int_\pi^0 \frac{1}{\delta e^{i\theta}} i\delta e^{i\theta}\,d\theta = -ia_{-1} \int_0^\pi d\theta = -\pi i a_{-1}$$

■

Example 6.32. Consider $f(z) = \frac{e^{iz}}{z}$. If $0 < \delta < R$, then

$$\left(\int_{-R}^{-\delta} + \int_\delta^R \right) f(x)\,dx = \left(\int_{-R}^{-\delta} + \int_\delta^R \right) \frac{\cos x + i \sin x}{x}\,dx = 2i \int_\delta^R \frac{\sin x}{x}\,dx$$

by even/oddness. Moreover, by Lemma 6.31,



$$\lim_{\delta \to 0} \int_{C_\delta} \frac{e^{iz}}{z}\,dz = -i\pi \operatorname*{Res}_{z=0} f(z) = -i\pi$$

Since $|f(z)| = \frac{e^{-y}}{R} \le \frac{1}{R}$ on C_R, Jordan's lemma tells us that

$$0 = 2i \int_0^\infty \frac{\sin x}{x}\,dx - i\pi \implies \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

The example relied on the evenness of $\frac{\sin x}{x}$ and on the fact that the region of the half-plane between C_R and C_δ contains no poles of $f(z)$. We essentially evaluated $\int_0^R \frac{\sin x}{x}\,dx = \frac{1}{2} \int_{-R}^R \frac{\sin x}{x}\,dx$ using an *indented path* lying on the x-axis but dodging round the simple pole at zero. Many other versions of this trick are possible!

Exercises 6.3 Many of these problems require extensive calculation to evaluate using residues: take your time and use it as an excuse to practice the previous section.

1. Use residues to verify the values of the improper integrals:

   (a) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4}$ (b) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^6+1} = \frac{\pi}{6}$

   (c) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$ (d) $\displaystyle\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$

2. Find the Cauchy principal value of the integrals:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^2+2x+2}$ (b) $\displaystyle\int_{-\infty}^\infty \frac{x\mathrm{d}x}{(x^2+1)(x^2+2x+2)}$

3. Let m,n be integers where $0 \le m \le n-2$. By mimicking Example 6.27.3, prove that

   $$\int_0^\infty \frac{x^m}{x^n+1}\,\mathrm{d}x = \frac{\pi}{n}\csc\frac{(m+1)\pi}{n}$$

4. (a) If $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges, prove that it equals its Cauchy principal value.

   (b) Suppose $f(x)$ is an even function and that P.V. $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists. Prove that $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ exists and has the same value.

5. If $f(x) = \frac{p(x)}{q(x)}$ is a rational function where $q(x)$ has no zeros and where $2+\deg p \le \deg q$, prove that $\int_0^\infty f(x)\,\mathrm{d}x$ converges.

   (*Hint: let p,q be monic and recall the comparison test for improper integrals*)

6. Prove the integration formulae:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$ if $a > b > 0$

   (b) $\displaystyle\int_0^\infty \frac{\cos ax\,\mathrm{d}x}{(x^2+b^2)^2} = \frac{\pi}{4b^3}(1+ab)e^{-ab}$ if $a,b > 0$

7. Evaluate the integrals:

   (a) $\displaystyle\int_{-\infty}^\infty \frac{x\sin x\,\mathrm{d}x}{(x^2+1)(x^2+4)}$ (b) $\displaystyle\int_0^\infty \frac{x^3\sin x\,\mathrm{d}x}{(x^2+1)(x^2+9)}$

8. If a is any real number and $b > 0$, find the Cauchy principal value of $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,\mathrm{d}x}{(x+a)^2+b^2}$

9. Use the function $f(z) = z^{-2}(e^{iaz} - e^{ibz})$ and an indented contour around $z_0 = 0$ to prove that

   $$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} = \frac{\pi}{2}(b-a) \quad a,b \ge 0$$

10. By integrating the function $f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp(-\frac{1}{2}\log z)}{z^2+1}$ where $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ along an indented contour, prove that

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

11. What happens to part 2 of Lemma 6.31 if $f(z)$ is analytic on $D \setminus \{z_0\}$ but has a pole of order $m \geq 2$ at z_0.

12. (Hard) A similar trick can be applied with sequences of boundary curves C_N. For instance, for each $N \in \mathbb{N}$, let C_N denote the positively oriented boundary of the square whose edges lie along the lines $x, y = \pm\left(N + \frac{1}{2}\right)\pi$. Prove that

$$\oint_{C_N} \frac{\mathrm{d}z}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

Hence conclude that $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$