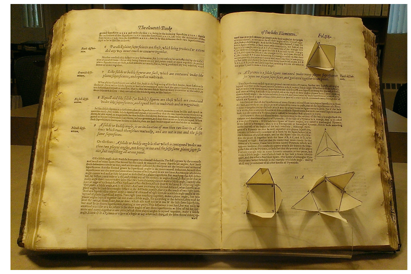
2 Euclidean Geometry

2.1 Euclid's Postulates and Book I of the Elements

Euclid's *Elements* (c. 300 BC) was a central part of the European and Islamic mathematical curriculum until the mid 20th century. Several examples are shown below.

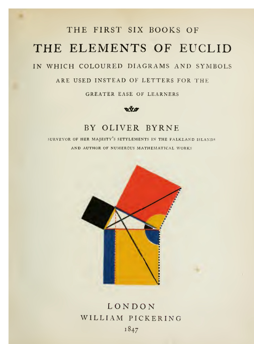
Earliest Fragment c. AD 100

Full copy, Vatican, 9th C

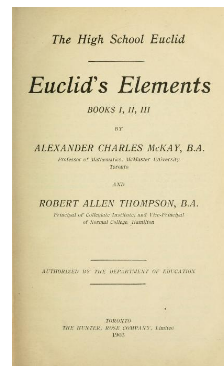


Pop-up edition 1500s

Latin translation 1572



Color edition 1847



Textbook 1903

Many of Euclid's arguments can be found online, and you can read Byrne's 1847 edition here.[5]

In this section we present an overview of Book I, which culminates in a proof of Pythagoras' Theorem (as seen on Byrne's cover). We begin by considering how Euclid's structure fits into the modern framework of an axiomatic system.

Undefined Terms E.g., point, line, etc. In fact Euclid attempted to define these: 'A point is that which has no part,' and 'A line has length but no breadth.'

Axioms/Postulates In Euclid, an axiom is somewhat more general than a postulate.

- A1 If two objects equal a third, then the objects are equal (= is transitive)
- A2 If equals are added to equals, the results are equal ($a = c \;\&\; b = d \implies a + b = c + d$)
- A3 If equals are subtracted from equals, the results are equal
- A4 Things that coincide are equal (in magnitude)
- A5 The whole is greater than the part

[5] To address some of the shortcomings in Euclid's original approach, Byrne contains a more comprehensive list of definitions, has more axioms, and relabels propositions 4 and 5 as axioms (pages xviii–xxiii).

Note the generality: the axioms seemingly refer to *any* objects. By contrast, Euclid's postulates contain the *geometry*: memorize these!

- P1 A pair of points may be joined to create a line
- P2 A line may be extended
- P3 Given a center and a radius, a circle may be drawn
- P4 All right-angles are equal
- P5 If a straight line crosses two others and the angles on one side sum to less than two right-angles then the two lines (when extended) meet on that side.

The first three postulates describe *ruler and compass constructions.* P4 allows Euclid to compare angles at different locations. P5 is usually called the *parallel postulate.*

Euclid's system doesn't quite fit the modern standard, but the rough structure is there. We'll consider several serious shortcomings in later sections. For now we clarify two issues and introduce some notation.

Segments To Euclid, a line had *finite* extent—we call such a (*line*) *segment.* The segment joining points A, B (postulate P1) is denoted $\overline{AB}$. In modern parlance a *line* extends as far as permitted, often infinitely.

Congruence Euclid uses *equal* where modern mathematicians say *congruent*. We'll express congruent segments and angles using modern notation: e.g., $\angle ABC \cong \angle DEF$. Equal angles/segments must be genuinely the same object (same location, etc.). otherwise said: equals implies congruence, but the converse is false.

Basic Theorems à la Euclid

Theorems are typically presented as a *problem*. Euclid provides a constructive solution (P1–P3), then proves that his construction really does solve the problem. Here is the first result of Book I.

Theorem 2.1 (I. 1). *Problem: to construct an equilateral triangle on a given segment.*

Note that a triangle is equilateral if its three sides are congruent.

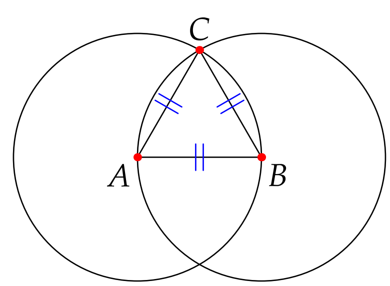
Proof. Given a line segment $\overline{AB}$:

By P3, construct circles centered at A and B with radius $\overline{AB}$.

Label one of the intersection points C. By P1, we may construct $\overline{AC}$ and $\overline{BC}$.

We claim that $\triangle ABC$ is equilateral.

Since $\overline{AB}$ and $\overline{AC}$ are radii of the circle centered at A, they are congruent. Similarly, $\overline{AB}$ and $\overline{BC}$ are congruent radii of the circle centered at B.
By Axiom A1, the three sides of $\triangle ABC$ are congruent.



■

Euclid proceeds to develop several well-known constructions and properties of triangles.

I. 4: Side-angle-side (SAS) congruence: if two triangles have two pairs of congruent sides and the angles between these are congruent, then the remaining sides and angles are also congruent in pairs.

$$\begin{cases} \overline{AB} \cong \overline{DE} \\ \angle ABC \cong \angle DEF \\ \overline{BC} \cong \overline{EF} \end{cases} \implies \begin{cases} \overline{AC} \cong \overline{DF} \\ \angle BCA \cong \angle EFD \\ \angle CAB \cong \angle FDE \end{cases}$$

I. 5: An isosceles triangle has congruent base angles.

I. 9: To bisect an angle.

I. 10: To find the midpoint of a segment.

I. 15: (Vertical angles) If two lines/segments intersect, opposite angles are congruent.

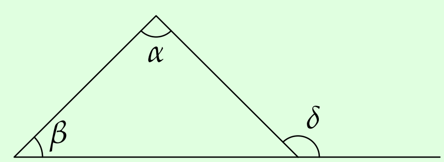
Despite various logical issues in Euclid's presentation, his arguments are still worth reading (try Byrne's edition). We'll revisit these basic results in the Exercises and in the next two sections.

Parallel Lines I: Construction & Existence

Definition 2.2. Lines are *parallel* if they do not intersect. Segments are parallel if no extensions of them intersect.

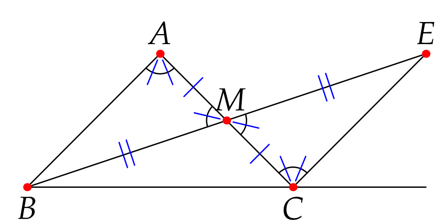
In Euclid, a line is not parallel to itself. The next result is one of the most important in Euclidean geometry, for it describes how to create a parallel line through a given point.

Theorem 2.3 (I. 16. Exterior Angle Theorem). *If one side of a triangle is extended, then the exterior angle is larger than either of the opposite interior angles.*
As pictured, $\delta > \alpha$ and $\delta > \beta$.



Euclid did not quantify angles numerically: $\delta > \alpha$ means α is congruent to some angle *inside* δ.

Proof. Construct the bisector $\overline{BM}$ of $\overline{AC}$ (I. 10).
Extend $\overline{BM}$ to E such that $\overline{BM} \cong \overline{ME}$ (I. 2).
Connect $\overline{CE}$ (P1).
The opposite angles at M are congruent (I. 15).
SAS (I. 4) applied to $\triangle AMB$ and $\triangle CME$ says $\angle BAM \cong \angle ECM$, which is smaller than the exterior angle at C.

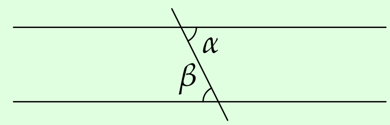


Bisect $\overline{BC}$ and repeat the argument to see that $\beta < \delta$: another invocation of the vertical angles theorem (I. 15) is needed, this time at C.

■

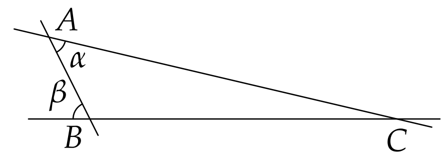
The proof in fact *constructs* a parallel ($\overline{CE}$) to $\overline{AB}$ through C, as the next result shows.

Theorem 2.4 (I. 27. Construction of a parallel).
If a line falls on two other lines such that the alternate angles (α, β) are congruent, then the two lines are parallel.



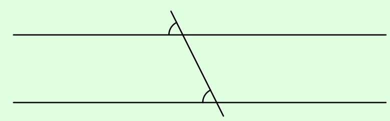
The *alternate angles* in the exterior angle theorem are those at A and C: $\overline{CE}$ really is parallel to $\overline{AB}$.

Proof. Assume, for contradiction, that the lines are not parallel.
WLOG suppose the lines meet on the right side at C.
By the Exterior Angle Theorem (I. 16), the angle β at B, being exterior to $\triangle ABC$, must be greater than the angle α at A: contradiction. ■



Euclid finishes the first half[6] of Book I by combining this construction with the vertical angles theorem (I. 15).

Corollary 2.5 (I. 28. Alternative construction of a parallel).
If a line falling on two other lines makes congruent angles, then the two lines are parallel.



Thus far Euclid has used only postulates P1–P4. In any model in which these hold:

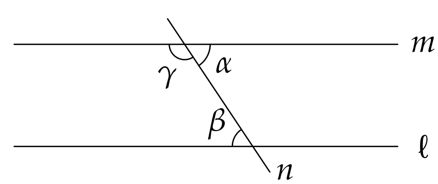
> Given a line ℓ and a point C not on ℓ, **there exists** a parallel to ℓ through C

Parallel Lines II: Uniqueness & Angle-sums

Euclid finally invokes the parallel postulate (P5) to prove that the congruent alternate angle approach is the *only* way to have parallel lines.

Theorem 2.6 (I. 29. Uniqueness of parallels—converse to I. 27).
If a line falls on two parallel lines, then the alternate angles are congruent.

Proof. We are given lines ℓ, m and a crossing line n.
Label the alternate angles α, β as in the picture, and let γ be the supplementary angle to α. We must prove that if ℓ and m are parallel, then $\alpha \cong \beta$.
Suppose to the contrary: WLOG we may assume $\alpha > \beta$.
But then $\beta + \gamma < \alpha + \gamma$, which is a straight-edge.
By the parallel postulate, ℓ and m meet on the left side of the picture: they are not parallel. ■

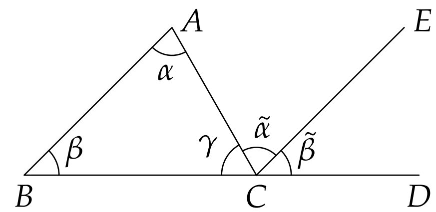


[6] Between I. 16 and 27 are several results regarding comparisons of sides and angles of triangles (the triangle inequality), and further triangle congruence theorems such as ASA. We'll consider these later.

The most well-known result about triangles is now within our grasp: the interior angles sum to a straight-edge. Euclid phrases this slightly differently.

Theorem 2.7 (I. 32. Angle sum in a triangle equals a straight-edge). *If one side of a triangle is extended, the exterior angle is congruent to the sum of the opposite interior angles.*

This is not a numerical sum, though for familiarity's sake we'll often write 180° for a straight-edge and 90° for a right-angle. In the picture we've labelled the angles for clarity. The result amounts to showing that $\tilde{\alpha} + \tilde{\beta} \cong \alpha + \beta$.



Proof. Construct $\overline{CE}$ as in I. 16 so that $\tilde{\alpha} \cong \alpha$. By I. 17, $\overline{CE}$ is parallel to $\overline{AB}$. $\overline{BD}$ falls on the parallels $\overline{AB}$ and $\overline{CE}$, whence $\tilde{\beta} \cong \beta$ (Corollary of I. 29). Axiom A2 shows that $\angle ACD = \tilde{\alpha} + \tilde{\beta} \cong \alpha + \beta$. ∎

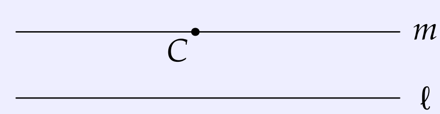
Since the three angles at C sum to a straight-edge the angle sum in the triangle is plainly

$$\alpha + \beta + \gamma \cong \tilde{\alpha} + \tilde{\beta} + \gamma \cong 180°$$

Playfair's Postulate

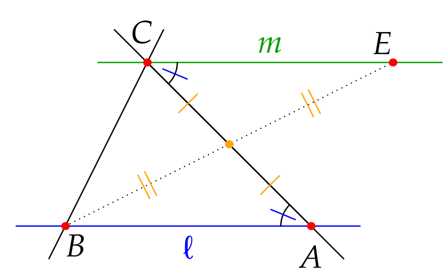
The parallel postulate is stated negatively (angles *don't* sum to a straight-edge, therefore lines are *not* parallel). Euclid likely chose this formulation to facilitate contradiction arguments, though it also obscures the meaning of the parallel postulate. Here is a more modern interpretation.

Axiom 2.8 (Playfair's Postulate).
Given a line ℓ and a point C not on ℓ, **at most one** parallel m to ℓ passes through C.



Our discussion thus far shows that the parallel postulate implies Playfair.

- Let $A, B \in \ell$ and construct the triangle $\triangle ABC$.
- The exterior angle theorem constructs E and thus a parallel m to ℓ by I. 27.
- I. 29 invokes the parallel postulate to prove that this is the *only* such parallel.



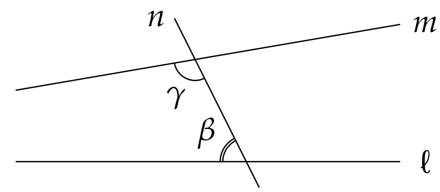
In fact the postulates are equivalent.

Theorem 2.9. *In the presence of Euclid's first four postulates, Playfair's postulate and the parallel postulate (P5) are equivalent.*

Proof. We proved that P5 implies Playfair above.

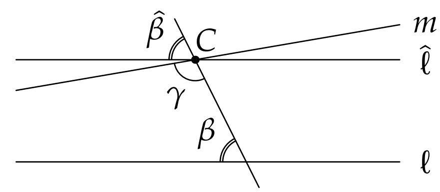
For Playfair ⇒ P5, we prove the contrapositive. Assume postulates P1–P4 are true and that P5 is *false.* Using quantifiers, and with reference to the picture, we restate the parallel postulate:[7]

P5: $\forall$ lines ℓ, m, n, $\beta + \gamma < 180° \implies \ell, m$ meet on the left side.



Its *negation* (P5 false) is therefore:

$\exists$ lines ℓ, m, n for which $\beta + \gamma < 180°$ and ℓ, m **do not meet** on the left side.



That ℓ, m are parallel is Exercise 5. By I. 28, we may build a *second* parallel line $\hat{\ell}$ to ℓ through the intersection C of m and n: as pictured,

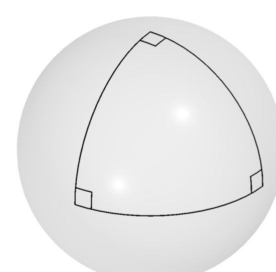
$$\hat{\beta} \cong \beta \implies \hat{\beta} + \gamma \cong \beta + \gamma < 180°$$

shows that $\hat{\ell} \neq m$. Crucially, these constructions only require postulates P1–P4.
We therefore have a point C not on a line ℓ, though which pass (at least) *two* parallels to ℓ: this is the negation of Playfair's postulate. ∎

Non-Euclidean Geometry

That Euclid waited so long before invoking the parallel postulate suggests he wanted to establish as much as possible in its absence. By contrast, everything from I. 29 onwards relies on the parallel postulate, including the proof that the angle sum in a triangle is 180°. For centuries, many mathematicians believed, though none could prove, that such a fundamental fact must be true independent of the parallel postulate.

Loosely speaking, a *non-Euclidean geometry* is a model for which a parallel through an off-line point either doesn't exist or is non-unique. It wasn't until the 17–1800s and the development of *hyperbolic geometry* (Chapter 4) that a model was found which satisfies Euclid's first four postulates and the negation of the parallel postulate.[8]



We shall eventually see that every triangle in hyperbolic geometry has angle sum less than 180°, though this will require a lot of work! For a more easily visualized non-Euclidean geometry consider the sphere. A rubber band stretched between three points on its surface describes a *spherical triangle*: an example with angle sum 270° is drawn. A similar game can be played on a saddle-shaped surface: as in hyperbolic geometry, 'triangles' will have angle sum less than 180°.

[7]Throughout, ℓ, m, n are distinct lines, with n intersecting both others. We strip this out for brevity.

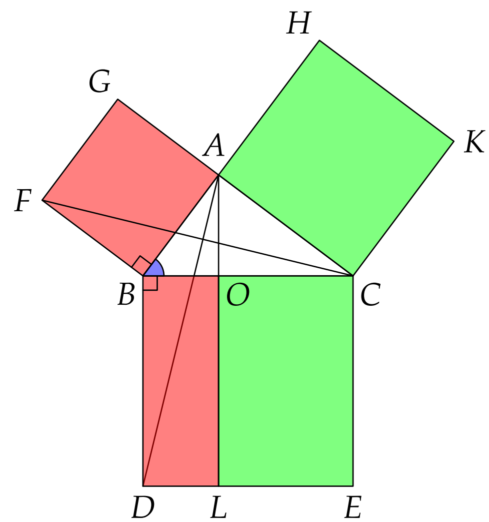
[8]The parallel postulate is therefore independent; in fact all Euclid's postulates are independent. They are also consistent (the 'usual' points and lines in the plane being a model), but incomplete: a sample undecidable is in Exercise 6.

Pythagoras' Theorem

Following his discussion of parallels, Euclid shows that parallelograms with the same base and height have equal area (I. 33–41), before providing constructions of parallelograms and squares (I. 42–46). Some of this is in Exercise 2. Immediately afterwards comes the capstone of Book I.

Theorem 2.10 (I. 47. Pythagoras' Theorem). *The square on the hypotenuse of a right triangle equals (has the same area as) the sum of the squares on the other sides.*

Proof. The given triangle $\triangle ABC$ is assumed to have a right-angle at A.



1. Construct squares on each side of $\triangle ABC$ (I. 46) and a parallel $\overline{AL}$ to $\overline{BD}$ (I. 16).
2. $\overline{AB} \cong \overline{FB}$ and $\overline{BD} \cong \overline{BC}$ since sides of squares are congruent. Moreover $\angle ABD \cong \angle FBC$ since both contain $\angle ABC$ and a right-angle.
3. Side-angle-side (I. 4) says that $\triangle ABD$ and $\triangle FBC$ are congruent (identical up to rotation by 90°).
4. I. 41 compares areas of parallelograms and triangles with the same base and height:

$$\begin{aligned}\text{Area}(\square ABFG) &= 2\,\text{Area}(\triangle FBC)\\ &= 2\,\text{Area}(\triangle ABD)\\ &= \text{Area}\left(\square BOLD\right)\end{aligned}$$

5. Similarly $\text{Area}(\square ACKH) = \text{Area}\left(\square OCEL\right)$.
6. Sum the rectangles to obtain $\square BCED$ and complete the proof. ∎

Euclid finishes Book I with the converse which we state without proof. The argument is sneaky, and typical of Euclidean geometry—look it up!

Corollary 2.11 (I. 48. Converse of Pythagoras'). *If the squares on two sides of a triangle equal the square on the third, then the triangle has a right-angle opposite the third side.*

The other twelve books of the *Elements* discuss further geometric constructions, including in three dimensions, alongside basic number theory including the Euclidean algorithm (Book VII).

While undoubtedly a masterpiece of logical reasoning, Euclid's presentation has several flaws. Most problematic is his reliance on pictorial reasoning: for instance, he 'proves' the SAS and SSS congruence theorems (I. 4 & 8) by laying one triangle on top of another, a process not justified by his axioms (see here or in Byrne). In a modern sense, Euclid's approach is part axiomatic system and part model: his reasoning requires a visual/physical representation of lines, circles, etc. In part because of these issues, we now turn to a more modern description of Euclidean geometry courtesy of David Hilbert.

Exercises 2.1. *Key concepts: Postulates P1–P5, Exterior angle theorem and* ***Existence*** *of parallels, Parallel postulate and* ***Uniqueness*** *of parallels, Playfair's postulate, Non-Euclidean geometry*

1. (a) Prove the *vertical angles theorem* (I. 15): if two lines cross, opposite angles are congruent.
   (*Hint: You will need postulate 4 regarding right-angles*)

   (b) Use part (a) to complete the proof of the exterior angle theorem: why is $\beta < \delta$?

   (c) Use the exterior angle theorem to prove that the sum of any two angles in a triangle is less than a straight-edge.
   (*This follows from I. 32, but that later result* ***requires uniqueness of parallels***)

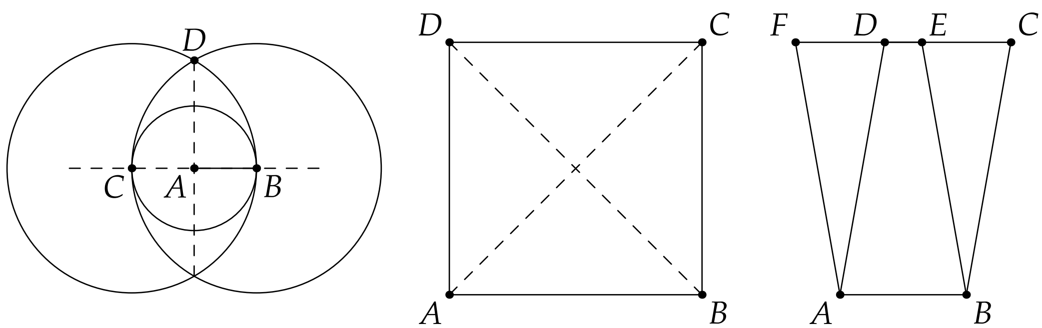
2. To help prove Pythagoras', Euclid makes use of the following results. Prove them as best as you can. Full rigor is tricky, but the pictures should help.

   (a) (I. 11) At a given point on a line, to construct a perpendicular.

   (b) (I. 46) To construct a square on a given segment.

   (c) (I. 35) Parallelograms on the same base and with the same height are equal (in area).

   (d) (I. 41) A parallelogram is twice (the area of) a triangle with the same base and height.



Theorem I. 11 Theorem I. 46 Theorem I. 35

3. In spherical geometry (page 16), *lines* are paths of shortest distance (great circles).

   (a) Which of Euclid's postulates P1–P5 are satisfied by this geometry?
   (*Some debate is allowed here, so offer your best reasoning*)

   (b) The exterior angle theorem is *false* in spherical geometry. Provide a counter-example, and explain where the proof fails.

4. Prove that Playfair's postulate is equivalent to the following statement:

   Whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other.

5. In the proof of Theorem 2.9, explain why ℓ, m must be parallel.
   (*You can't use I. 32 for this; why not?*)

6. The *line-circle continuity* property states:

   If P lies inside and Q outside a circle α, then the segment $\overline{PQ}$ intersects α.

   By considering the rational points in the plane $\mathbb{Q}^2 = \{(x, y) : x, y \in \mathbb{Q}\}$, show that the line-circle continuity property is undecidable within Euclid's system.

7. The standard proof of the converse of Pythagoras' Theorem (I. 48) is a *corollary* of the original! Look it up and explain the argument as best you can.

2.2 Hilbert's Axioms I: Incidence and Order

The long process of identifying and correcting the errors and omissions in Euclid's *Elements* arguably culminated in David Hilbert's *Grundlagen der Geometrie* (*Foundations of Geometry*) (1899). In the next two sections we consider some of the details of Hilbert's approach, thus providing a modern description of Euclidean geometry.

A version of Hilbert's axioms for plane geometry are listed on the next page.[9] The undefined terms consist of two types of object (*points* and *lines*), and three relations (*between* $*$, *on* $\in$ and *congruence* $\cong$). For brevity we'll often (ab)use set notation, viewing a line as a set of points, though this is not necessary. At various places, definitions and notations are required.

Definition 2.12. Throughout, A, B, C denote distinct points and ℓ, m lines.

Line: $\overleftrightarrow{AB}$ denotes the line through distinct A, B. This exists and is unique by axioms I-1 and I-2.

Segment: $\overline{AB} := \{A, B\} \cup \{C : A * C * B\}$ has *endpoints* A, B and all *interior* points C between them.

Ray: $\overrightarrow{AB} := \overline{AB} \cup \{C : A * B * C\}$ has *vertex* A. In essence we extend $\overline{AB}$ beyond B.

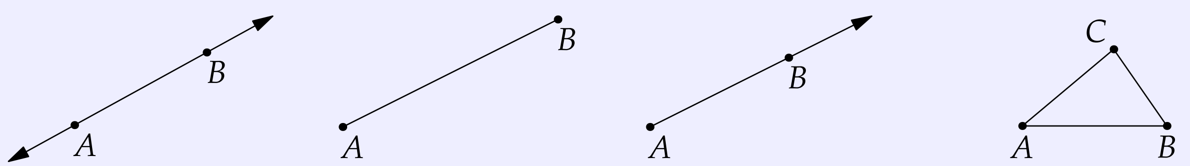
Triangle: $\triangle ABC := \overline{AB} \cup \overline{BC} \cup \overline{CA}$ where A, B, C are non-collinear. Triangles are *congruent* if their sides and angles are congruent in pairs.

Sides of a line: A, B (not on ℓ) lie on the *same side* of ℓ if $\overline{AB} \cap \ell = \emptyset$. Otherwise A and B lie on *opposite sides* of ℓ.

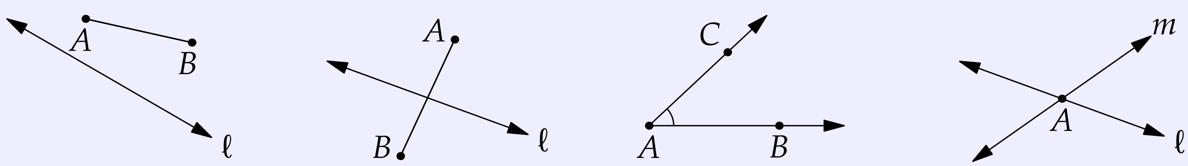
Angle: $\angle BAC := \overrightarrow{AB} \cup \overrightarrow{AC}$ has *vertex* A and *sides* $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Parallelism: ℓ and m *intersect* if some point lies on both: $\exists A \in \ell \cap m$. Lines are *parallel* if they do not intersect. Segments/rays are parallel when the corresponding lines are parallel.

The pictures represent these notions in the usual model of Cartesian geometry.



Line $\overleftrightarrow{AB}$ Segment $\overline{AB}$ Ray $\overrightarrow{AB}$ Triangle $\triangle ABC$



Same side Opposite sides Angle $\angle BAC$ Intersection $A \in \ell \cap m$

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[9] Like Euclid, Hilbert also covered 3D geometry, though we only give the axioms for plane geometry. With regard to Definition 1.6, Hilbert's system is about as good as can be. Essentially one only one model exists (almost the same thing as completeness). The system is consistent in the absence of the continuity axiom though, in line with Gödel (Theorem 1.7), consistency cannot be proved once continuity is included. As stated, the axioms are not quite independent, though this can be remedied: O-3 does not require existence (follows from Pasch's axiom), C-1 does not require uniqueness (follows from uniqueness in C-4) and C-6 can be weakened slightly.

Hilbert's Axioms for Plane Geometry

Undefined terms

1. *Points*: use capital letters, $A, B, C \dots$
2. *Lines*: use lower case letters, $\ell, m, n, \dots$
3. *On*: $A \in \ell$ is read 'A lies on ℓ' or 'ℓ contains A'
4. *Between*: $A * B * C$ is read 'B lies between A and C'
5. *Congruence*: $\cong$ is a binary relation on segments or angles

Axioms of Incidence

I-1 For any distinct A, B *there exists* a line ℓ on which lie A, B.

I-2 Distinct A, B lie on *at most* one common line.

Notation: *line* $\overleftrightarrow{AB}$ through/containing A and B

I-3 On every line there exist at least two distinct points. There exist at least three non-collinear points.

Axioms of Order

O-1 If $A * B * C$, then A, B, C are distinct points on the same line, and $C * B * A$.

O-2 Given distinct A, B, there is at least one point C such that $A * B * C$.

O-3 If A, B, C are distinct points on the same line, exactly one lies between the others.

Definitions: *segment* $\overline{AB}$ and *triangle* $\triangle ABC$

O-4 (Pasch's Axiom) Suppose none of A, B, C lie on ℓ. If some point of ℓ lies on the segment $\overline{AB}$, then ℓ also contains a point of $\overline{AC}$ or $\overline{BC}$.

Definitions: *sides* of line $\overleftrightarrow{AB}$ and *ray* $\overrightarrow{AB}$

Axioms of Congruence

C-1 (Segment transference) Let A, B be distinct and r a ray based at A'. Then there exists a unique point $B' \in r$ for which $\overline{AB} \cong \overline{A'B'}$. Moreover $\overline{AB} \cong \overline{BA}$.

C-2 If $\overline{AB} \cong \overline{EF}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{CD}$.

C-3 If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

Definition: *angle* $\angle ABC$

C-4 (Angle transference) Given $\angle BAC$ and $\overrightarrow{A'B'}$, there exists a unique ray $\overrightarrow{A'C'}$ on a given side of $\overleftrightarrow{A'B'}$ for which $\angle BAC \cong \angle B'A'C'$.

C-5 If $\angle ABC \cong \angle GHI$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle DEF$. Moreover, $\angle ABC \cong \angle CBA$.

C-6 (Side-angle-side) Given triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$, then the triangles are congruent.[10]

Axiom of Continuity Suppose a line/ray/segment ℓ is partitioned into non-empty subsets Σ_1, Σ_2 such that no point of one lies between two of the other. Then there exists a unique point O separating Σ_1, Σ_2: for all $A, B \in \ell$,

$A * O * B$ if and only if A, B lie in distinct subsets $\Sigma_1 \setminus O$, $\Sigma_2 \setminus O$

Playfair's Axiom Definition: *parallel lines*

Given a line ℓ and a point $P \notin \ell$, at most one line through P is parallel to ℓ.

[10] Its sides/angles are congruent in pairs. We extend congruence to other geometric objects similarly.

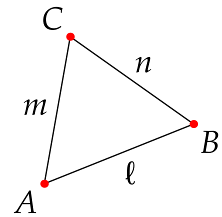
Axioms of Incidence: Finite Geometries

The axioms of incidence describe the relation *on*: 'A point lies *on* a line/segment/ray.' An *incidence geometry* is any model satisfying axioms I-1, I-2 & I-3. Perhaps surprisingly, there exist incidence geometries with *finitely many* points!

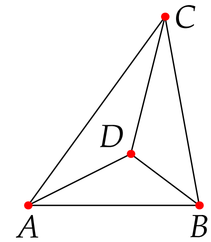
Examples 2.13. By I-3, an incidence geometry requires *at least three* points.

Up to relabelling, a unique 3-point geometry exists: I-3 says A, B, C are non-collinear; by I-1 and I-2, each pair lies on a unique line, whence there are precisely three lines

$$\ell = \{A, B\}, \quad m = \{A, C\}, \quad n = \{B, C\}$$



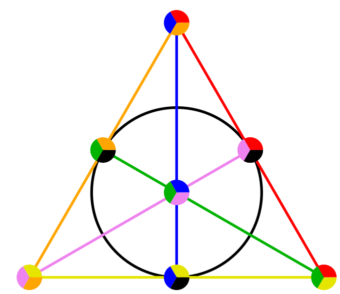
3 points, 3 lines



4 points, 6 lines

Up to relabelling, there are two incidence geometries with four points: one is drawn; how many lines has the other?

The final picture is a seven-point incidence geometry called the *Fano plane,* which finds many applications particularly in combinatorics. Each point lies on precisely three lines and each line contains precisely three points—each dot is colored to indicate the lines to which it belongs. Don't be fooled by the black line looking 'curved' and seeming to cross the blue line near the top, for the line only contains three points!



We can even prove some simple theorems in incidence geometry. The second is an exercise.

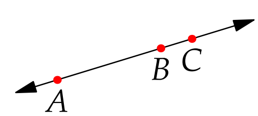
Lemma 2.14. *If distinct lines intersect, then they do so in exactly one point.*

Proof. Suppose A, B are distinct points of intersection. By axiom I-2, there is at most one line through A and B. Contradiction. ∎

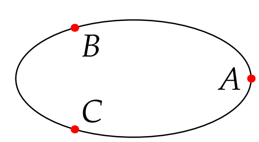
Lemma 2.15. *Given any point, there exist at least two lines on which it lies.*

Axioms of Order: Sides of a Line, Pasch's Axiom & the Crossbar Theorem

The order axioms describe the ternary relation *between*. The first three clarify the idea that points on a line lie in a fixed order: travelling along a line, one encounters each point exactly once. In particular, these axioms prevent 'circular' lines (the pictured contradiction) and guarantee that a line contains infinitely many points.



$A * B * C$ ✓



Contradiction

Each of the above finite incidence examples fails to satisfy at least one order axioms: is it obvious which?

The rest of this section is devoted to the consequences of Pasch's axiom (O-4)—named for Moritz Pasch (c. 1882). Amongst other things, it helps demonstrate that the *interior* of various objects are non-empty.

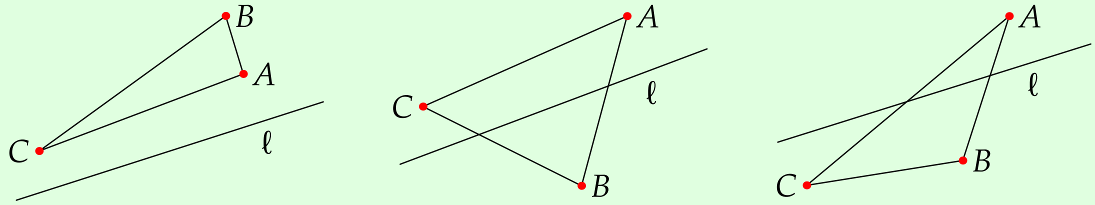
Lemma 2.16 (Exercise 5). *Every segment contains an interior point.*

By inducting on the Lemma, every segment contains *infinitely many* points.

Sidedness We now establish that a line has precisely two *sides* (Definition 2.12). This concept lies behind several of Euclid's arguments without being properly defined in the *Elements*.

Theorem 2.17 (Plane Separation). *A line ℓ separates all points not on ℓ into two half-planes: the two* ***sides*** *of ℓ. More explicitly, suppose A, B, C are distinct points which do not lie on ℓ. Then:*

1. *If A, B lie on the* ***same*** *side of ℓ and B, C lie on the* ***same*** *side, then A, C lie on the* ***same*** *side.*
2. *If A, B lie on* ***opposite*** *sides and B, C lie on* ***opposite*** *sides, then A, C lie on the* ***same*** *side.*
3. *If A, B lie on* ***opposite*** *sides and B, C lie on the* ***same*** *side, then A, C lie on* ***opposite*** *sides.*



Case 1 *Case 2* *Case 3*

Proof. We prove the contrapositive of case 1. Suppose A, B, C are non-collinear. If $\overline{AC}$ intersects ℓ, then ℓ intersects one side of $\triangle ABC$. By Pasch's axiom, it also intersects either $\overline{AB}$ or $\overline{BC}$.

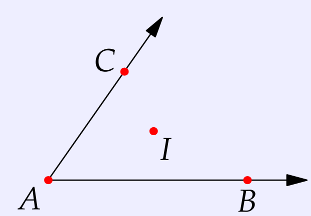
The other cases are exercises, and we omit the tedious collinear possibilities. ∎

Plane separation and sidedness allow us to properly define interiors of angles and triangles.

Definition 2.18. A point I is *interior* to $\angle BAC$ if is lies on the same side of $\overleftrightarrow{AB}$ as C and same side of $\overleftrightarrow{AC}$ as B.

Otherwise said, I lies in the intersection of two half-planes.

A point I is *interior* to $\triangle ABC$ if it is interior to all three angles $\angle ABC$, $\angle BAC$ and $\angle ACB$. Otherwise said, I lies in the triple intersection of three half-planes defined by the triangle's sides.



Interior points permit comparison of angles sharing a vertex: if I is interior to $\angle BAC$, then $\angle BAI < \angle BAC$ has obvious meaning *without resorting to numerical angle measure.*

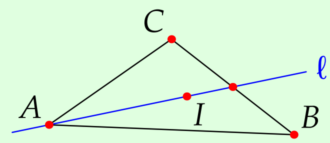
Corollary 2.19. *Every angle has an interior point. Every triangle has an interior point.*

Proof. Given $\angle BAC$, consider any interior point I of $\overline{BC}$. This plainly lies on the same side of $\overleftrightarrow{AB}$ as C and on the same side of $\overleftrightarrow{AC}$ as B. Exercise 8 completes the argument for a triangle. ∎

The Crossbar Theorem To paraphrase Pasch's axiom: *if a line enters a triangle, it must come out.* We haven't quite established this fact, however. What if the line passes through a vertex?

Theorem 2.20. *If I is interior to $\angle BAC$, then $\overrightarrow{AI}$ intersects $\overline{BC}$ (the "crossbar") at an interior point.*

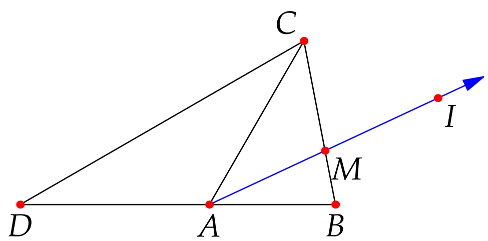
The result specializes to triangles. Suppose a line ℓ passes through a vertex A and at least one interior point I of $\triangle ABC$: then the line also contains an interior point of the side $\overline{BC}$ opposite the vertex.



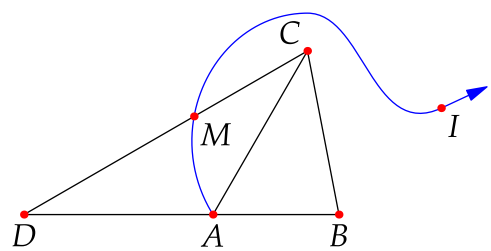
Proof. Extend $\overline{AB}$ to D such that $B * A * D$ (O-2). Since C is not on $\overleftrightarrow{BD} = \overleftrightarrow{AB}$, we have $\triangle BCD$. Since $\overleftrightarrow{AI}$ intersects one edge of $\triangle BCD$ at A and does not cross any vertices (think about why...), Pasch says $\overleftrightarrow{AI}$ intersects one another edge ($\overline{BC}$ or $\overline{CD}$) at some point M.

The result follows from applying plane separation to the lines $\overleftrightarrow{AB} = \overleftrightarrow{BD}$ and $\overleftrightarrow{AC}$. First observe:

> Since I, M lie on the same side of $\overleftrightarrow{AB} = \overleftrightarrow{BD}$ as C, it follows that $\overline{IM}$ does not intersect $\overleftrightarrow{AB}$. Since A, I, M are collinear and $A \in \overleftrightarrow{AB}$, we moreover have $A \notin \overline{IM}$.



correct arrangement[11]



contradictory arrangement

If $M \in \overline{BC}$, we are done. For contradiction, suppose $M \in \overline{CD}$. Relative to $\overleftrightarrow{AC}$:

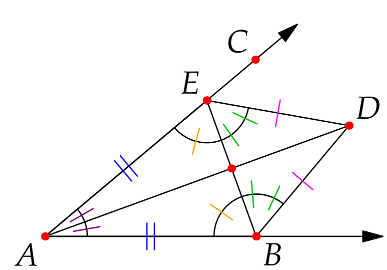
- I and B lie on the same side since I is interior to $\angle BAC$;
- B and D lie on opposite sides, since $B * A * D$ and $\overleftrightarrow{AC} \neq \overleftrightarrow{BD} = \overleftrightarrow{AB}$;
- D and M lie on the same side since $M \in \overline{CD}$ and $\overleftrightarrow{CD} \neq \overleftrightarrow{AC}$.

By plane separation, I, M lie on opposite sides of $\overleftrightarrow{AC}$. The collinearity of A, I, M then forces the contradiction $A \in \overline{IM}$. ■

Euclid repeatedly uses the crossbar theorem without justification, including in his construction of perpendiculars and angle/segment bisectors (Theorems I. 9 & 10). We sketch the latter here.

Given $\angle BAC$, construct E such that $\overline{AB} \cong \overline{AE}$. Construct D using an equilateral triangle (I. 1). SSS (I. 8) shows that $\angle BAC$ is bisected, and SAS (I. 4) that $\overrightarrow{AD}$ bisects $\overline{BE}$.

Quite apart from Euclid's arguments for SAS and SSS being suspect (we'll deal with these in the next section), he gives no argument for why D is interior to $\angle BAC$ or why $\overrightarrow{AD}$ should intersect $\overline{BE}$!



[11]The pictures could be modified: e.g., $I = M$ and $A * I * M$ are also correct arrangements ($M \in \overline{BC}$).

Even with Pasch's axiom and the crossbar theorem, it requires some effort to repair Euclid's argument. This is ultimately of no consequence: once congruence has been introduced, Hilbert provides an alternative construction of the bisector once congruence has been introduced.

Exercises 2.2. *Key concepts: Incidence axioms (meaning of* ***on****), Finite incidence geometries, Order axioms (meaning of* ***between****), Pasch's Axiom, Sides of a line, Crossbar theorem*

1. Label the vertices in the Fano plane 1 through 7 (any way you like). As we did in Example 2.13 for the 3-point geometry, describe each line in terms of its points.

2. Prove Lemma 2.15.

3. (a) Give a model for each of the 5-point incidence geometries. How many are there? (*Hint: order doesn't matter, so the only issue is how many points lie on each line*...)

   (b) It is possible for there to be a 6-point incidence geometry so that each line contains precisely three points? Why/why not?

4. Consider the proof of the crossbar theorem (2.20). How can we be certain that $\overleftrightarrow{AI}$ does not contain any of the vertices of $\triangle BCD$.

5. We prove Lemma 2.14, that every segment has an interior point.

   Let A, B be distinct points. Using only the axioms of incidence and order (and Lemma 2.14 which follows from I-2), demonstrate the existence of each of the pictured points C, D, E, F *in alphabetical order.* Point F proves the Lemma.

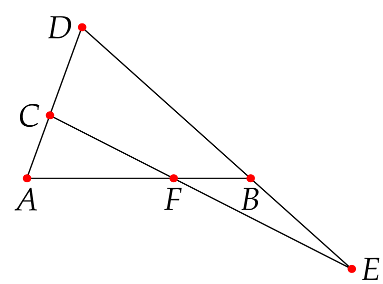
   During your construction, make sure to address the following issues:

   (a) Explain why D does not lie on $\overleftrightarrow{AB}$.

   (b) Explain why E does not lie on $\triangle ABD$.

   (c) Explain why $E \neq C$ (whence $\overleftrightarrow{CE}$ exists).

   (d) Explain why F lies on $\overline{AB}$ and *not* on $\overline{BD}$.

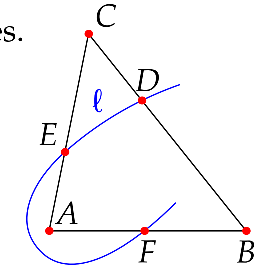


6. We complete the proof of the plane separation theorem (2.17).

   (a) Prove part 3 (it is almost a verbatim application of Pasch's axiom).

   (b) Suppose a line ℓ intersects all three sides of $\triangle ABC$ but no vertices. This results in a very strange picture (we've labelled the intersections D, E, F and WLOG chosen $D * E * F$).

   Apply Pasch's axiom to $\triangle DBF$ and $\overleftrightarrow{AC}$ to obtain a contradiction. Hence establish part 2 of the plane separation theorem.



7. Suppose A, B, C are distinct points on a line ℓ.

   (a) Explain why there exists a line $m \neq \ell$ such that $B \in m$.

   (b) Prove that $A * B * C$ if and only if A and C lie on opposite sides of m.

   (c) Suppose $A * B * C$. Use part (b) to prove the following:

      i. B is the only point common to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

      ii. If $D \in \ell$ is any point other than B, then D lies in precisely one of $\overrightarrow{BA}$ or $\overrightarrow{BC}$.

8. Prove that the interior of a triangle is non-empty.

   (*Hint: use Exercise 5 to construct a suitable I, then prove that it lies on the correct side of each edge*)

9. The existence of infinitely many points on a line follows easily from the fact that every segment has an interior point. Find an alternative proof that does not depend on Pasch's axiom.

   (*Hint: you will need the other order axioms!*)

2.3 Hilbert's Axioms II: Congruence

Hilbert's congruence axioms address two primary issues in Euclid.

1. Euclid uses of *equal(s)* is confusing. In Hilbert, segments/angles are equal only when they are precisely the same (this amounts to the *reflexivity* part of the next result).

2. Euclid makes frequent unjustified uses of pictorial reasoning. For instance, his "proof" of the SAS and SSS triangle congruence theorem (I. 4) is little more that the claim that if one triangle is "laid on top of the other," then they match up. It was eventually realized that one of the triangle congruences has to be an axiom: SAS is Hilbert's axiom C-6.

We start with a small piece of bookkeeping.

Lemma 2.21. *Congruence of segments/angles is an equivalence relation.*

Proof. We prove the result for segments, leaving the statement for angles as an exercise.

Reflexivity: Let $\overline{AB}$ be given. Apply C-1 to obtain $\overline{A'B'}$ such that $\overline{AB} \cong \overline{A'B'}$. We sneakily use this *twice* and apply C-2 to obtain

$$\overline{AB} \cong \overline{A'B'} \text{ and } \overline{AB} \cong \overline{A'B'} \implies \overline{AB} \cong \overline{AB}$$

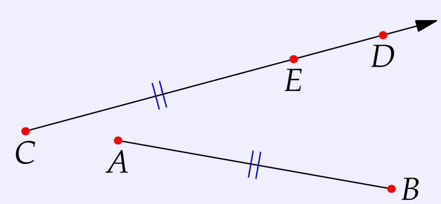
Symmetry: Assume $\overline{AB} \cong \overline{CD}$. By reflexivity, $\overline{CD} \cong \overline{CD}$. By C-2 we have $\overline{CD} \cong \overline{AB}$.

Transitivity: Suppose $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$. By symmetry, $\overline{EF} \cong \overline{CD}$. Axiom C-2 now shows that $\overline{AB} \cong \overline{EF}$. ∎

Segment/Angle Transfer and Comparison

Hilbert's axioms of segment and angle transference are crucial for comparing non-collinear segments, and angles at different locations. These axioms help clean up Euclid's unjustified arguments based on pictorial reasoning.

Definition 2.22. Let segments $\overline{AB}$ and $\overline{CD}$ be given.
Axiom C-1 says there is a unique point E on the ray $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$: we have *transferred* $\overline{AB}$ onto $\overrightarrow{CD}$.
We write $\overline{AB} < \overline{CD}$ if E lies between C and D (pictured); similarly, if $C * D * E$, then $\overline{AB} > \overline{CD}$.



By O-3, any two segments are comparable: given $\overline{AB}$ & $\overline{CD}$, precisely one of the following holds,

$$\overline{AB} < \overline{CD}, \qquad \overline{CD} < \overline{AB}, \qquad \overline{AB} \cong \overline{CD}$$

Moreover, axiom C-3 says that congruence respects 'addition' of adjacent congruent segments. Unique angle transfer, comparison and addition follow similarly from axiom C-4 and Definition 2.18 (interior points).

Neither Hilbert nor Euclid use or require an *absolute* notion of length or angle-measure: the comparison $\overline{AB} < \overline{CD}$ does not indicate a relationship between numerical quantities (lengths). Introducing numerical length requires the inclusion of the real numbers (and thus far more axioms)—for purity reasons, we postpone this until later in this chapter.

The Triangle Congruence Theorems: SAS, ASA, SSS & SAA

Hilbert assumes side-angle-side (SAS) as an axiom (C-6) and proceeds to prove the remaining results. Here is ASA: we'll cover SSS momentarily, and SAA is Exercise 5.

Theorem 2.23 (Angle-Side-Angle, Euclid I. 26, case I). *Suppose $\triangle ABC$ and $\triangle DEF$ satisfy*

$$\angle ABC \cong \angle DEF, \qquad \overline{AB} \cong \overline{DE}, \qquad \angle BAC \cong \angle EDF$$

Then the triangles are congruent $(\angle ACB \cong \angle DFE,\ \overline{AC} \cong \overline{DF}$ *and* $\overline{BC} \cong \overline{EF})$.

It is important to see that the triangle congruence theorems are independent of Playfair's Axiom (uniqueness of parallels). As such, we cannot simply assert $\angle ACB \cong \angle DFE$ and apply SAS.

Hilbert's approach modifies Euclid's: instead of laying $\triangle ABC$ on top of $\triangle DEF$, he creates a new triangle $\triangle DEG \cong \triangle ABC$ and proves that $G = F$.

Proof. Segment transfer provides the unique point $G \in \overrightarrow{EF}$ such that $\overline{EG} \cong \overline{BC}$.

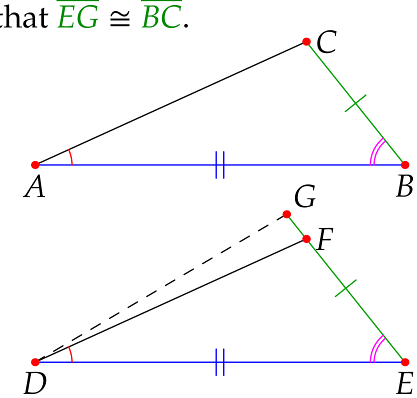
Apply SAS to $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEG$ $(= \angle DEF)$, $\overline{BC} \cong \overline{EG}$, to see that $\angle BAC \cong \angle EDG$.

By assumption and the fact that congruence is an equivalence relation, $\angle EDG \cong \angle BAC \cong \angle EDF$.

Since F and G lie on the same side of $\overleftrightarrow{DE}$, angle transference (C-4) says they lie on the *same ray* through D.

But then F and G both lie on two distinct lines ($\overleftrightarrow{EF} = \overleftrightarrow{EG}$ and $\overleftrightarrow{DF} = \overleftrightarrow{DG}$). We conclude that $F = G$.

By SAS we conclude the remaining data: $\triangle ABC \cong \triangle DEF$.



∎

Geometry Without Circles

Circles are at the heart of Euclid's constructions (the *compass* part of *ruler-and-compass* constructions). Since circle intersections depend on continuity, Hilbert essentially ignores them. In the remainder of this section, we sketch a few of his alternative approaches to Euclid's basic results.

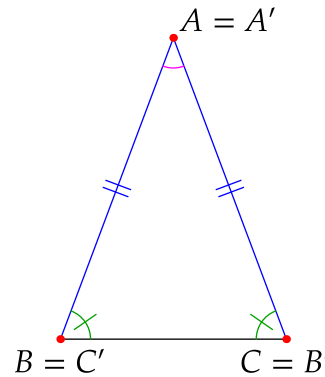
Theorem 2.24 (Euclid I. 5). *An isosceles triangle has congruent base angles.*

Isosceles means *equal legs*: two sides of the triangle are assumed congruent. The remaining side is the *base.* Euclid relies on a famously complicated construction (look it up!). Hilbert does things more sneakily by relabelling the original triangle and applying SAS.

Proof. Suppose $\triangle ABC$ is isosceles where $\overline{AB} \cong \overline{AC}$.

Define a 'new' triangle $\triangle A'B'C' = \triangle ACB$ by switching the base points:

$$A' := A, \qquad B' := C, \qquad C' := B \qquad (*)$$



By axiom C-5, angles with reversed legs are congruent:

$$\angle BAC \cong \angle CAB$$

Relabelling the right side of this and both sides of our original assumption $\overline{AB} \cong \overline{AC}$, we obtain

$$\overline{AB} \cong \overline{A'B'}, \quad \angle BAC \cong \angle B'A'C', \quad \overline{AC} \cong \overline{A'C'}$$

We now have SAS! In conclusion, $\angle ABC \cong \angle A'B'C'$ is congruent to $\angle ACB$ by relabeling. ■

Is Hilbert's argument really all that different from Thales' (page 2)? All he's done is to 'reflect' the original triangle, but in a manner consistent with the axioms!

Theorem 2.25 (Euclid I. 12). *To drop a perpendicular from a given point to a line.*

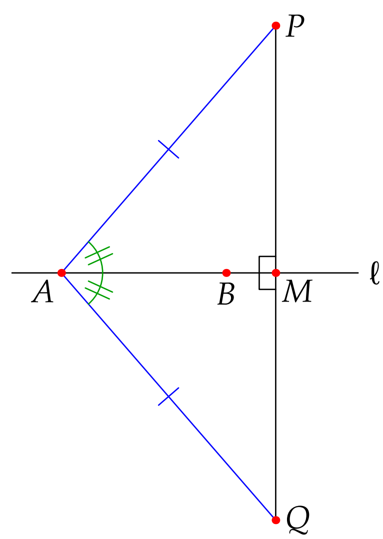
Euclid accomplishes this using circle intersections and SSS.[12] Hilbert sticks with segment/angle transference and sidedness before appealing to SAS (axiom C-6).

Proof. Suppose P dos not lie on ℓ. Our goal is to construct $M \in \ell$ such that $\overline{PM}$ meets ℓ in a right-angle.

Let A, B be distinct points on ℓ (axiom I-3) so that $\ell = \overleftrightarrow{AB}$.

By axioms C-4 and C-1, transfer $\overline{AP}$ to the other side of ℓ at A, creating a new point Q.

Since P and Q lie on opposite sides of ℓ, the line intersects $\overline{PQ}$ at some point M. There are two cases to consider.



- In the generic case $M \neq A$ (pictured), SAS applied to $\triangle MAP$ and $\triangle MAQ$ shows that $\angle AMP \cong \angle AMQ$. Since these angles sum to a straight-edge ($\overline{PQ}$), they are both right-angles.
- In the extreme case $M = A$, there are no triangles and SAS cannot be applied. Instead, observe that B does not lie on $\overline{PQ}$ (which axioms/results make this clear?!) and apply the above argument with B instead of A.

■

Generalizing this construction facilitates a corrected argument for the SSS triangle congruence.

[12]Consider a picture similar to Thm. I. 11 in Exercise 2.2.2.

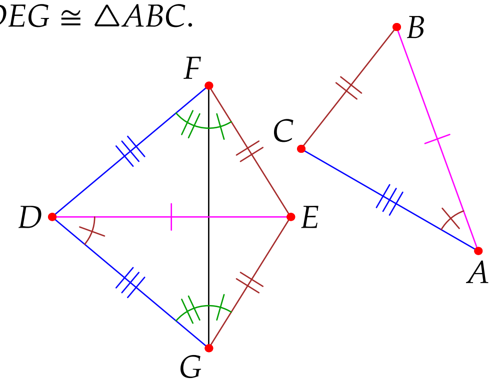
Theorem 2.26 (Side-Side-Side, Euclid I. 8). *If $\triangle ABC$ and $\triangle DEF$ have sides congruent in pairs,*

$$\overline{AB} \cong \overline{DE}, \qquad \overline{BC} \cong \overline{EF}, \qquad \overline{AC} \cong \overline{DF}$$

then the triangles are congruent ($\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, $\angle CAB \cong \angle FDE$).

The strategy is similar to the proof of ASA: create a new triangle $\triangle DEG \cong \triangle ABC$...

Proof. Transfer $\angle BAC$ to D on the opposite side of $\overleftrightarrow{DE}$ to F, creating G (axioms C-4 and C-1). SAS ($\overline{AB} \cong \overline{DE}$, $\angle BAC \cong \angle EDG$, $\overline{AC} \cong \overline{DG}$) shows that $\triangle DEG \cong \triangle ABC$.



Join $\overline{FG}$ to produce isosceles triangles $\triangle FDG$ and $\triangle FEG$ each with base $\overline{FG}$: both therefore have congruent base angles at F and G. Sum these angles and apply SAS

$$\left.\begin{array}{l} \overline{DF} \cong \overline{DG} \\ \angle DFE \cong \angle DGE \\ \overline{EF} \cong \overline{EG} \end{array}\right\} \implies \triangle DEF \cong \triangle DEG$$

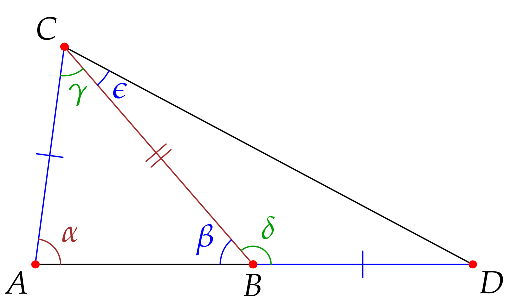
We conclude that $\triangle ABC \cong \triangle DEG \cong \triangle DEF$, as required.[13] ■

Exterior Angle Theorem (Thm. 2.3, I. 16) Hilbert ignores Euclid's bisector approach.

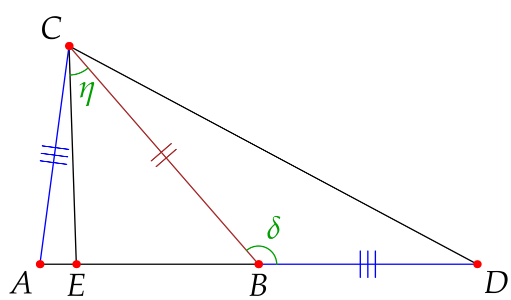
Proof. Given $\triangle ABC$, extend $\overline{AB}$ to D such that $\overline{AC} \cong \overline{BD}$. As pictured, we prove that the alternatives to $\gamma < \delta$ are impossible (angles are labeled with Greek letters for clarity).

1. ($\delta \not\cong \gamma$) Assume $\delta \cong \gamma$. By SAS, $\triangle ACB \cong \triangle DBC$; in particular $\epsilon \cong \beta$. Since A and D lie on opposite sides of $\overleftrightarrow{BC}$, we see that $\epsilon + \gamma \cong \beta + \delta$ is a straight-edge. But then A, D are distinct points lying on *two* lines! Contradiction.

2. ($\delta \not< \gamma$) Assume $\delta < \gamma$. Transfer δ to C as shown to obtain $\eta \cong \delta$ inside γ. By the crossbar theorem, we obtain the intersection point E. But now δ is an exterior angle to $\triangle EBC$ congruent to an opposite interior angle η of the same triangle, contradicting part 1.



Step 1: $\delta \cong \gamma$ is a contradiction



Step 2: $\delta < \gamma$ is a contradiction

Take the vertical angle to δ at B and repeat the argument to see that $\alpha < \delta$. ■

[13]To be formal, we should also deal with the situations where the sum is a subtraction.

Is Euclid now fixed?

Almost! In the exercises we show how the following may be achieved:

- Construction of an isosceles triangle on a segment $\overline{AB}$. With this, segment and angle bisectors can be constructed (Euclid I. 9+10).
- SAA congruence (Euclid I. 26, case II), the last remaining triangle congruence theorem.

With this, almost all of the first half of Book I (prior to the application of the parallel postulate): surprisingly, all that's missing is Euclid's first theorem on the construction of equilateral triangles! Once we include Playfair's axiom, the remainder of the book may be proved—from equilateral triangles to Pythagoras'—all *without circles!*

Exercises 2.3. *Key concepts: Segment & angle transfer (existence & uniqueness), Segment comparison, Triangle congruence theorems (SAS, ASA, SSS, SAA), Fixing Euclid's pictorial reasoning*

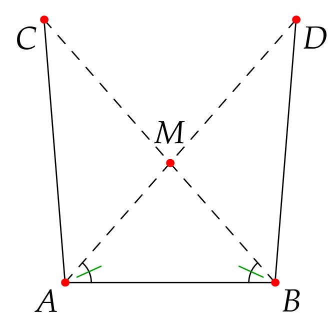
Except for question 8, all exercises should be answered without reference to continuity, circles, or the uniqueness of parallels (e.g., Playfair's axiom, (tri)angle sum $\cong 180°$).

1. Draw pictures to suggest why you don't expect Angle-Angle-Angle (AAA) and Side-Side-Angle (SSA) to be triangle congruence theorems.

2. Use axioms C-4 and C-5 to prove that congruence of angles is an equivalence relation.

3. (a) Suppose a triangle has two angles congruent. Use ASA to prove that it is isosceles.
   (b) Explain why the base angles of an isosceles triangle are *acute* (less than a right-angle).

4. Given $\overline{AB}$, axiom I-3 says $\exists C \notin \overleftrightarrow{AB}$.
   If $\triangle ABC$ is not isosceles, then WLOG assume $\angle ABC < \angle BAC$. Transfer $\angle ABC$ to A to produce D on the same side of $\overleftrightarrow{AB}$ as C with

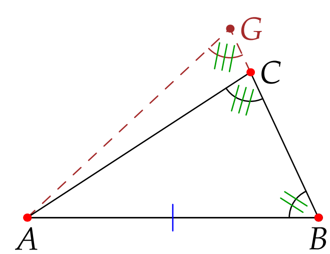
   $$\angle ABC \cong \angle BAD, \qquad \overline{BC} \cong \overline{AD}$$

   (a) Explain why rays $\overrightarrow{AD}$ and $\overrightarrow{BC}$ intersect (at some point M).
   (b) Why is $\triangle MAB$ isosceles?
   (c) Describe how to produce the perpendicular bisector of $\overline{AB}$.
   (d) Explain how to construct an angle bisector using the above discussion.

5. Prove the SAA congruence. If $\triangle ABC$ and $\triangle DEF$ satisfy

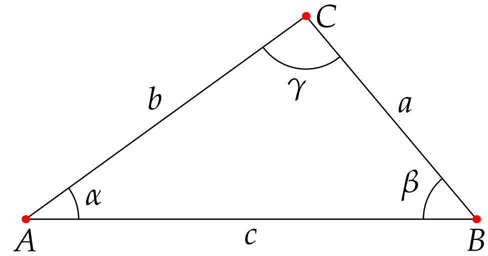
   $$\overline{AB} \cong \overline{DE}, \quad \angle ABC \cong \angle DEF \quad \text{and} \quad \angle BCA \cong \angle EFD$$

   then the triangles are congruent: $\triangle ABC \cong \triangle DEF$.
   (*Hint: Let $G \in \overrightarrow{BC}$ be such that $\overline{BG} \cong \overline{EF}$...*)

6. We prove Theorems I. 18, 19 and 20 on comparisons of angles and sides in a triangle.

   For clarity, suppose $\triangle ABC$ has sides and angles as labelled in the picture.

   (a) (I. 18) Assume $a < c$. Prove that $\alpha < \gamma$.
   (*Hint: let D on $\overline{AB}$ satisfy $\overline{BD} \cong a$*)

   (b) (I. 19: converse to I. 18) $\alpha < \gamma \Longrightarrow a < c$.
   (*Hint: Prove the contrapositive*)

   (c) (I. 20: triangle inequality) $a + b > c$.
   (*Hint: Let E lie on $\overrightarrow{BC}$ such that $\overline{CE} \cong b$ and apply I. 19*)

7. Hilbert's published SAS axiom is weaker than we've stated:

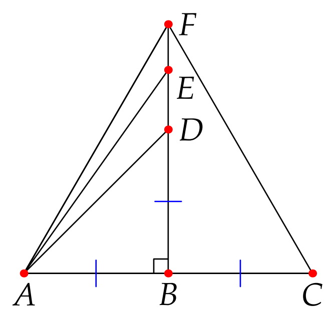
   $$\text{If } \overline{AB} \cong \overline{A'B'}, \overline{AC} \cong \overline{A'C'} \text{ and } \angle BAC \cong \angle B'A'C', \text{ then } \angle ABC \cong \angle A'B'C'.$$

   Use this to prove the full SAS congruence theorem (axiom C-6).

   (*Hint: try a trick similar to the proof of ASA*)

8. Construct the picture on the right, where $\overline{BF}$ is the perpendicular bisector of $\overline{AC}$, and

   $$\overline{BD} \cong \overline{AB}, \qquad \overline{BE} \cong \overline{AD}, \qquad \overline{BF} \cong \overline{AE}$$

   Use Pythagoras' Theorem to prove that $\triangle ACF$ is equilateral.

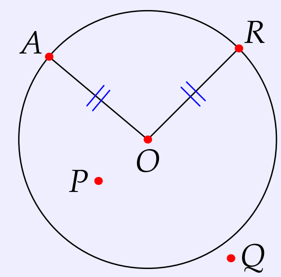
   This construction requires Playfair's axiom and thus unique parallels. Unlike Euclid's proof of Theorem 2.1 (I. 1), it does not require circle intersections (continuity). As we'll see later, Theorem I. 1 can alternatively be proved by assuming all of Hilbert's axioms except *Playfair.*

2.4 Circles and Continuity

Circles were entirely natural to ancient cultures and could be created using a very simple machine (a peg and cord). As such, their intersections are central to Euclid's presentation. They are also very easy to define formally: all that's required is segment congruence and, if we want to define the inside and outside, segment comparison.

Definition 2.27. Let O and R be distinct points. The *circle* $\mathcal{C}$ with center O and radius $\overline{OR}$ is the collection of points A such that $\overline{OA} \cong \overline{OR}$.

Since all segments are comparable (Definition 2.22), every point satisfies exactly one of three possibilities:



- A lies *on* $\mathcal{C}$ (that is, $\overline{OA} \cong \overline{OR}$).
- P lies *inside* $\mathcal{C}$ if $P = O$ or $\overline{OP} < \overline{OR}$.
- Q lies *outside* $\mathcal{C}$ if $\overline{OR} < \overline{OQ}$.

Otherwise said: a circle partitions the plane into three subsets.

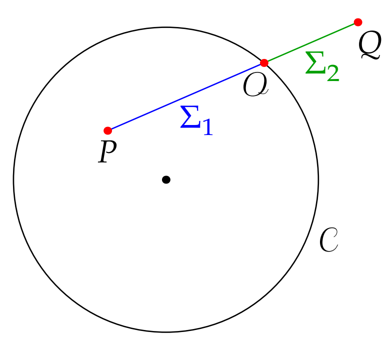
It seems obvious from the picture that if we want to travel from the inside of a circle to the outside, then at some point we must be *on* the circle. Euclid relies on this "obvious" principle without justification. The modern *Axiom of Continuity* provides exactly the necessary justification. This axiom is more technical than the others. As such, it is natural for Hilbert barely to mention circles; instead he builds as much geometry as he can using only the simpler axioms.

Two facts must be established in order to correct Euclid's approach.

Theorem 2.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be circles.*

1. *(Elementary/Line-Circle Continuity Principle) If P lies inside and Q lie outside $\mathcal{C}$, then the segment $\overline{PQ}$ intersects $\mathcal{C}$ at exactly one point.*
2. *(Circular Continuity Principle) Suppose $\mathcal{D}$ contains two points, one inside and the other outside $\mathcal{C}$. Then the circles intersect in precisely two points. The intersection points moreover lie on* opposite sides *of the line through the circle centers.*

The idea of the first principle is to partition $\overline{PQ}$ into two pieces:



Σ_1 consists of the points lying on or inside $\mathcal{C}$.

Σ_2 consists of the points lying outside $\mathcal{C}$.

One proves that Σ_1 and Σ_2 satisfy the hypotheses of the continuity axiom. The resulting point $\mathcal{O}$ is then shown to lie on $\mathcal{C}$ itself. Some of the details are in Exercise 7.

The circular continuity principle is harder.

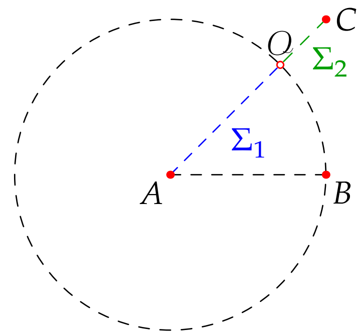
Example 2.29. To convince ourselves why the continuity axiom is necessary, it is helpful to consider a model for which it is false. For ease of understanding, we use the language of co-ordinates.

Rational geometry $\mathbb{Q}^2 = \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$ consists of those points in the Cartesian plane with rational co-ordinates. It satisfies almost all of Hilbert's axioms: C-1 and continuity are false. To see why, consider the picture, and the points with co-ordinates $A = (0,0)$, $B = (1,0)$, $C = (1,1)$ and $\mho = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. It should be clear that $\overline{AB} \cong \overline{A\mho}$ (both segments have length 1).

Axiom C-1 Within $\mathbb{Q}^2$, we cannot transfer $\overline{AB}$ to the ray $\overrightarrow{AC}$, since $\mho$, having irrational co-ordinates, is not a point in the geometry.

Continuity Consider the circle centered at A with radius 1. Plainly A lies inside this circle and C outside. However the segment $\overline{AC}$ does not intersect the circle, thus contradicting the elementary continuity principle.

More properly, $\overline{AC} = \Sigma_1 \cup \Sigma_2$ may be partitioned as shown and yet no point $\mho$ in the geometry separates Σ_1, Σ_2.



Equilateral triangles

Armed with circle intersections, we can finally correct Euclid's very first proof!

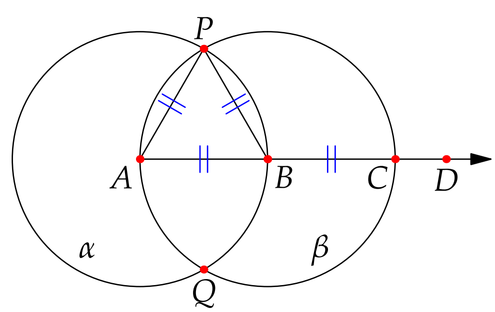
Theorem 2.30 (Euclid I.1). *An equilateral triangle many be constructed on a given segment $\overline{AB}$.*

Proof. Following Euclid, consider the circles α and β centered respectively at A and B, both having radius $\overline{AB}$. We apply two of Hilbert's axioms and then the circular continuity principle.

Axiom O-2: $\exists D$ such that $A * B * D$.

Axiom C-1: Let $C \in \overrightarrow{BD}$ be the unique point for which $\overline{BC} \cong \overline{AB}$.

Circular Continuity principle: β contains A (inside α) and C (outside α). The circles therefore intersect in two points P and Q.



The rest of the argument is essentially Euclid's. Since P lies on both circles (and is therefore distinct from A and B), we have $\overline{AP} \cong \overline{AB}$ and $\overline{BP} \cong \overline{AB}$: the three sides are congruent and so $\triangle ABC$ is equilateral.[14] ∎

As we saw in Exercise 2.3.8, if Playfair's axiom regarding unique parallels is included, the above can be proved without circles or the continuity axiom. Regardless, we can finally say that every result in Book I of Euclid is correct, even if his original axioms and arguments are insufficient!

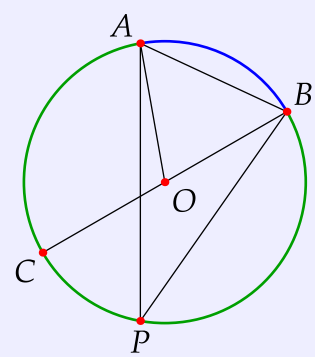
[14] Strictly, this last step uses Lemma 2.21 and thus axioms C-1 and C-2, though it is not really possible to *define* an equilateral triangle without it!

Basic Circle Geometry

We continue our survey of Euclidean geometry with a few results about circles. Many of these are found in Book III of the *Elements*. For the rest of this chapter, we assume all of Hilbert's axioms including Playfair and Continuity; what follows often relies on their consequences, particularly angle-sums in triangles and the circular continuity principle.

Definition 2.31. With reference to the picture:

- A *chord* $\overline{AB}$ is a segment joining two points on a circle.
- A *diameter* $\overline{BC}$ is a chord passing through the center O.
- $\angle AOB$ is a *central* angle and $\angle APB$ an *inscribed* angle.
- $\triangle ABP$ is *inscribed* in its *circumcircle.*
- An *arc* $\overset{\frown}{AB}$ is part of the circular edge between chord points (major or minor by length[15]).



The following ideas should not be new, so most of the details are left as exercises.

Theorem 2.32 (III. 20). *If $\triangle ABP$ is inscribed in a circle such that $\overset{\frown}{AB}$ is a minor arc, then the central angle is twice the inscribed angle: $\angle AOB = 2\angle APB$.*

For a sketch proof, join $\overline{OP}$, breaking $\triangle ABP$ into three isosceles triangles before counting angle sums. A little extra is required if the center O lies outside the triangle.

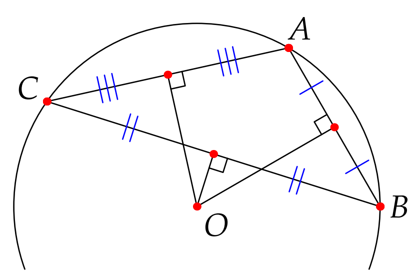
Corollary 2.33. 1. (*III. 21*) *If inscribed triangles share an arc, then the inscribed angles are congruent.*

2. (*III. 22*) *An inscribed quadrilateral has opposite angles supplementary* (*summing to 180°*).

3. (*III. 31—Thales' Theorem*) *A triangle in a semi-circle is right-angled.*

Theorem 2.34. *Every triangle has a unique circumcircle.*

This is similar to III. 1: construct the perpendicular bisectors of two sides as in the picture; necessarily these meet at the center of the required circle, though there is still something to prove...



[15] A minor arc corresponds to a (central) *angle*: in pure Euclidean geometry, all angles are positive and less than or equal to a straight-edge.

Our final results consider the concept of *tangency*. At first thought, you might suppose this to require some calculus, but it doesn't...

Definition 2.35. A line is *tangent* to a circle if it intersects the circle exactly once.

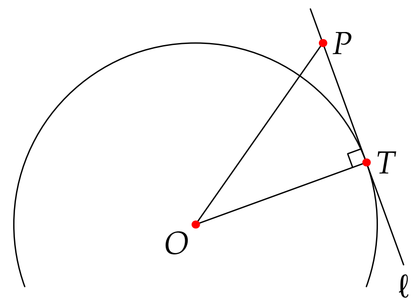
Theorem 2.36 (III. 18, 19 (in part)). *A line is tangent to a circle if and only if it is perpendicular to the radius at an intersection point.*

Proof. We prove the ($\Leftarrow$) direction, leaving the other half as an exercise.

Suppose T lies on a circle C and that ℓ passes through T perpendicular to the radius $\overline{OT}$.

Let P be any another point on ℓ. Since $\triangle OTP$ is right-angled, Exercise 2.3.6 (angle-side comparisons) tells us that

$$\angle OPT < 90^\circ \cong \angle OTP \implies \overline{OT} < \overline{OP}$$



That is, P lies *outside* the circle.
Otherwise said, T is the unique intersection point $\ell \cap C$, whence ℓ is tangent to the circle. ∎

Theorem 2.37. *Any given point outside a circle lies on exactly two lines tangent to the circle.*

If you can give formal proofs of all these results, you should consider yourself in a very good place regarding basic Euclidean geometry.

Exercises 2.4. *Key concepts: Continuity implies circle intersections, Inscribed & Central angles, Circumcircle, Line-Circle Tangency*

1. Prove all parts of Corollary 2.33.

2. Prove Theorem 2.34.

3. Given a circle centered at O and a point P outside the circle, draw the circle centered at the midpoint of $\overline{OP}$ passing through both O and P.

   Explain why the intersections of these circles are the points of tangency required in Theorem 2.37. Hence complete its proof.

4. (a) Prove Theorem 2.32 when O is *interior* to $\triangle ABP$.

   (b) Prove Theorem 2.32 when O is *exterior* to $\triangle ABP$.

5. Given $\triangle ABC$, prove that there exists exactly one circle (the *incircle*) which is tangent to all three sides.

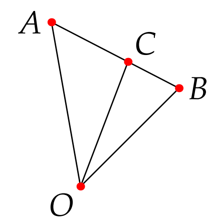
   (*Hint: Bisect two of the angles...*)

6. (a) Complete the proof of Theorem 2.36 by showing the ($\Rightarrow$) direction. Does this result require the continuity axiom?

   (*Hint: if T is an intersection and the angle isn't 90°, drop a perpendicular from O to ℓ*)

   (b) Suppose a line contains a point inside a circle. Prove that it intersects the circle in exactly two points. What about this time: do you need continuity now?

7. Suppose $A * C * B$ and that $O \notin \overleftrightarrow{AB}$. Use Exercise 2.3.6 to show that



$$\overline{OC} < \max(\overline{OA}, \overline{OB})$$

   If A, B are interior to a circle centered at O, conclude that C is also.

   (*This is part of what's needed to demonstrate the elementary continuity principle: no point of Σ_2 lies between two points of Σ_1. Can you prove the other condition?*)

8. (Hard) Suppose $A_0, A_1, B \in \ell$ satisfy $A_0 * A_1 * B$. Transfer $\overline{A_0A_1}$ repeatedly to create a sequence of points $A_0, A_1, A_2, A_3, \ldots$ as shown.

   We use the continuity axiom to prove the *Archimedean property*:

$$\exists N \text{ for which } A_0 * B * A_N$$

   Define $\Sigma_1 = \bigcup_{n=1}^{\infty} \overrightarrow{A_nA_0}$. Our goal is to prove that $\Sigma_1 = \ell$ (hence $B \in \Sigma_1$). For contradiction, suppose instead that $\Sigma_2 := \ell \setminus \Sigma_1$ is non-empty. Plainly $\ell = \Sigma_1 \cup \Sigma_2$.

   (a) Prove that no point $Y \in \Sigma_2$ lies between two points $X_1, X_2 \in \Sigma_1$.

   (b) Prove that no point $X \in \Sigma_1$ lies between two points $Y_1, Y_2 \in \Sigma_2$.

   (c) By parts (a,b), $\ell = \Sigma_1 \cup \Sigma_2$ satisfies the hypotheses of the continuity axiom. Let $\mathcal{O}$ be the unique point so defined. Obtain a contradiction by transferring $\overline{A_0A_1}$ to $\mathcal{O}$.

2.5 Similar Triangles, Length and Trigonometry

In pure Euclidean geometry there are *no numerical measures* of length or angle. Since segments and angles can be compared (Definition 2.22), we do have *relative* measure. For convenience we've denoted right-angles and straight-edges by 90° & 180° respectively. To avoid continued frustration, it is time we introduced explicit numerical measures, though to do so properly requires more axioms.

Axioms 2.38 (Length and Angle (Degree) Measure).

L1 To each segment $\overline{AB}$ corresponds a unique positive real number $|AB|$, its *length*.

L2 $|AB| = |CD| \iff \overline{AB} \cong \overline{CD}$.

L3 $|AB| < |CD| \iff \overline{AB} < \overline{CD}$.

L4 If $A * B * C$, then $|AB| + |BC| = |AC|$.

A1 To each angle $\angle ABC$ corresponds a unique real number $m\angle ABC$ between 0 and 180, its *degree measure*.

A2 $m\angle ABC = m\angle DEF \iff \angle ABC \cong \angle DEF$.

A3 $m\angle ABC < m\angle DEF \iff \angle ABC < \angle DEF$.

A4 If P is interior to $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$.

A5 Right-angles measure 90°.

In axioms L3 and A3, comparison of segments and angles has the meaning arising from the congruence axioms (Definition 2.22, etc.). It is not worth memorizing these axioms, just observe how they fit your intuition. Angle measure in Euclidean geometry has two notable differences from what you might expect:

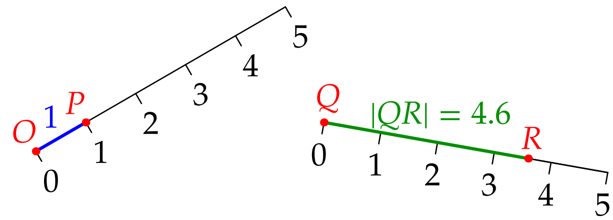
- (A1) All angles measure strictly between 0° and 180°. In particular, a straight-edge isn't an angle (though such is denoted 180°) and there are no *reflex angles* (> 180°).
- (A2) Angles are *non-oriented*, measuring the same with reversed legs ($m\angle ABC = m\angle CBA$).

The axioms for length and angle follow the same pattern except that A5 explicitly fixes the scale of angle measure. To do the same for length requires a choice of some reference segment of length 1. The following is a consequence of the continuity axiom.

Theorem 2.39 (Uniqueness of measure).

1. *Given $\overline{OP}$, there is a unique way to assign a length to every segment such that $|OP| = 1$.*
2. *There is a unique way to assign a degree measure to every angle.*

The segment $\overline{OP}$ in part 1 essentially provides the length-scale for a *ruler.* We measure the length of any segment by moving this ruler on top of the desired segment (segment-transference).



Area Measure

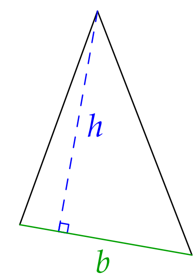
Including Playfair's axiom allows us to define *rectangles* (Exercise 2.1.2).

Definition 2.40. The *area* (*measure*) of a rectangle is the product of its base and height (measures).

Given a length measure, a square with side 1 necessarily has area 1.

Relative to a base segment, the *height* of a triangle is its *altitude*: the length of the perpendicular dropped from the opposite vertex.

Since every rectangle is a parallelogram, and every triangle half a parallelogram, Euclid's discussion (e.g., Thm. I. 35) amounts to the familiar area formulæ:



$$\text{area(parallelogram)} = bh, \qquad \text{area(triangle)} = \frac{1}{2}bh$$

While these expressions are nice to have, they are not necessary for what follows.

Lemma 2.41. *If triangles have congruent bases, then their areas are in the same ratio as their heights. The same holds with the roles of heights and bases reversed.*

Length and area were non-numerical quantities to the ancient Greeks, so Euclid worked with length and area *ratios*, an unnecessarily confusing approach for modern readers.[16] To us, the Lemma is simply division of real numbers.

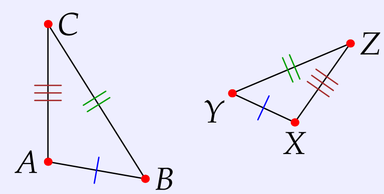
Euclid	Modern
$A_1 : A_2 = h_1 : h_2$	$\frac{A_1}{A_2} = \frac{\frac{1}{2}bh_1}{\frac{1}{2}bh_2} = \frac{h_1}{h_2}$

Similarity and the AAA Theorem

Similar triangles are the concern of Book VI of the *Elements.* All of this requires unique parallels.

Definition 2.42. Triangles are *similar,* written $\triangle ABC \sim \triangle XYZ$, if their sides are in the same length ratio

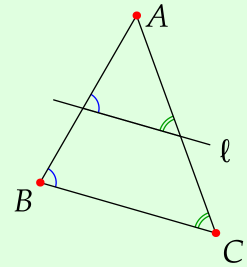
$$\frac{|AB|}{|XY|} = \frac{|BC|}{|YZ|} = \frac{|CA|}{|ZX|}$$



[16] Arguably the most difficult parts of the *Elements* are where Euclid grapples with what irrational ratios should mean (Books V & X).

Our primary result comes in two equivalent versions.

Theorem 2.43 (Angle-Angle-Angle/AAA, Euclid VI. 2–5).

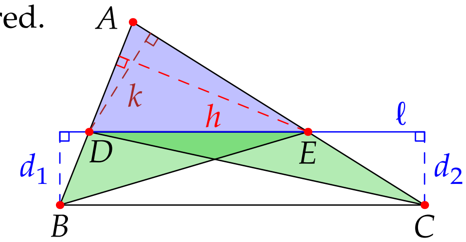


1. *Triangles are similar if and only if their angles form congruent pairs.*
2. *Suppose a line ℓ intersects two sides $\overline{AB}$ and $\overline{AC}$ of a triangle at D and E respectively. Then*

$$\triangle ADE \sim \triangle ABC \iff \ell \text{ is parallel to } \overleftrightarrow{BC}$$

The above picture should convince you that 1 $\Rightarrow$ 2 (uniqueness of parallels, Corollary 2.5, Theorem 2.6, etc.).[17] The converse of the equivalence (2 $\Rightarrow$ 1) is left as an Exercise. We therefore restrict ourselves to proving part 2.

Proof. ($\Leftarrow$) Drop perpendiculars to create h, k, d_1, d_2 as pictured.



Exercise 4 says ℓ is parallel to $\overline{BC}$ if and only if $d_1 = d_2$. By Lemma 2.41, triangles with the same height have areas proportional to their bases:

$$\frac{|BD|}{|AD|} = \frac{\text{area}(BDE)}{\text{area}(ADE)} \qquad (\triangle BDE,\ \triangle ADE \text{ share height } h)$$

$$\frac{|CE|}{|AE|} = \frac{\text{area}(CDE)}{\text{area}(ADE)} \qquad (\triangle CDE,\ \triangle ADE \text{ share height } k)$$

Since $\triangle BDE$ and $\triangle CDE$ share base $\overline{DE}$, we see that

$$\ell \text{ is parallel to } \overline{BC} \iff d_1 = d_2 \iff \text{area}(BDE) = \text{area}(CDE)$$
$$\iff \frac{|BD|}{|AD|} = \frac{|CE|}{|AE|} \qquad (*)$$

Add $1 = \frac{|AD|}{|AD|} = \frac{|AE|}{|AE|}$ to both sides to obtain one part of the required similarity ratio

$$\frac{|AB|}{|AD|} = \frac{|AD| + |BD|}{|AD|} = \frac{|AE| + |CE|}{|AE|} = \frac{|AC|}{|AE|}$$

It remains to see that this ratio equals $\frac{|BC|}{|DE|}$. Again using common heights (h, k) of triangles,

$$\frac{|AB|}{|BD|} = \frac{\text{area}(ABE)}{\text{area}(BDE)} \qquad \frac{|CE|}{|AE|} = \frac{\text{area}(BCE)}{\text{area}(ABE)} \qquad (\dagger)$$

$$\Longrightarrow \frac{|AB|}{|AD|} \overset{(*)}{=} \frac{|AB|}{|BD|}\frac{|CE|}{|AE|} \overset{(\dagger)}{=} \frac{\text{area}(BCE)}{\text{area}(BDE)} = \frac{|BC|}{|DE|}$$

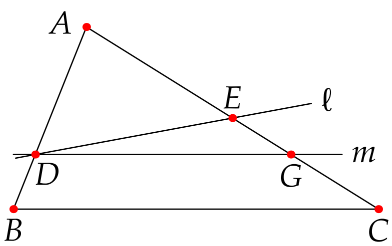
where the last equality follows since $\triangle BCE$ and $\triangle BDE$ have common height $d_1 = d_2$.

[17]This reliance on unique parallels is crucial: we do not expect AAA similarity in non-Euclidean geometry. Indeed we'll see in Chapter 4 that AAA is a *congruence* theorem in hyperbolic geometry!

(⇒) Suppose $\triangle ABC \sim \triangle ADE$.

Let m be the unique parallel to $\overline{BC}$ through D. This intersects $\overline{AC}$ at a point G. We must prove that $G = E$ (consequently $m = \ell$).

By the (⇐) direction above,



$$\triangle ABC \sim \triangle ADG$$

However $\sim$ is transitive (it is an equivalence relation), whence $\triangle ADE \sim \triangle ADG$. The similarity ratio is 1 ($|AD| : |AD|$), whence

$$\frac{|AE|}{|AG|} = 1 \implies |AE| = |AG| \implies E = G$$

■

Applications of Similarity

We finish with several advanced applications of similarity that were not known to Euclid.

Trigonometry Early trigonometry dates to a few centuries after Euclid, though the approach was different.[18]

Definition 2.44. Given an acute angle $\angle ABC$ ($m\angle ABC < 90°$), drop a perpendicular from A to $\overrightarrow{BC}$ at D so that $\angle ADB$ is a right-angle. Define

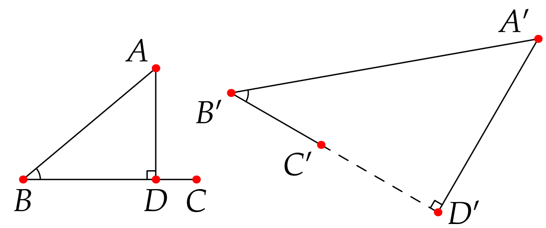
$$\sin \angle ABC := \frac{|AD|}{|AB|} \qquad \cos \angle ABC := \frac{|BD|}{|AB|}$$

Theorem 2.45. *Angles have the same sine (cosine) if and only if they are congruent.*

Proof. Assume $\angle ABC \cong \angle A'B'C'$, as pictured, and drop perpendiculars to D, D'.

Since $\triangle ABD$ and $\triangle A'B'D'$ have two pairs of mutually congruent angles, the third pair is congruent also. AAA therefore applies: the triangles are similar and

$$\frac{|AD|}{|AB|} = \frac{|A'D'|}{|A'B'|} \qquad \frac{|BD|}{|AB|} = \frac{|B'D'|}{|A'B'|}$$



In particular, $\sin \angle ABC = \sin \angle A'B'C'$ and $\cos \angle ABC = \cos \angle A'B'C'$.

The converse is an exercise.

■

[18] Ancient forerunners of sine and cosine were defined using chords of circles rather than triangles. The term *trigonometry* (literally *triangle measure*) wasn't coined until 1595.

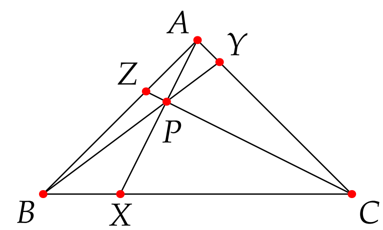
Cevians After Giovanni Ceva (1647–1734), a *cevian* is any segment joining a vertex to the opposite side of a triangle. Here is a result from the height of Euclidean geometry—good luck trying to prove it using co-ordinates!

Theorem 2.46 (Ceva's Theorem). *Given $\triangle ABC$ and cevians $\overline{AX}, \overline{BY}, \overline{CZ}$,*

$$\frac{|BX|}{|XC|}\frac{|CY|}{|YA|}\frac{|AZ|}{|ZB|} = 1 \iff \text{the cevians meet at a common point}$$

Proof. Suppose the cevians meet at P, and apply Lemma 2.41 twice.

$$\frac{\text{area}(ABX)}{\text{area}(AXC)} = \frac{|BX|}{|XC|} = \frac{\text{area}(PBX)}{\text{area}(PXC)}$$



A little algebra (exercise) shows that

$$\frac{|BX|}{|XC|} = \frac{\text{area}(ABX) - \text{area}(PBX)}{\text{area}(AXC) - \text{area}(PXC)} = \frac{\text{area}(ABP)}{\text{area}(APC)}$$

Repeat for the other ratios and multiply to get 1.

The converse is also an exercise. ∎

The Butterfly Theorem We finish with a beautiful result from roughly 1803–5. Several proofs are known; ours relies on similar triangles and a simple lemma whose proof is left as an exercise.

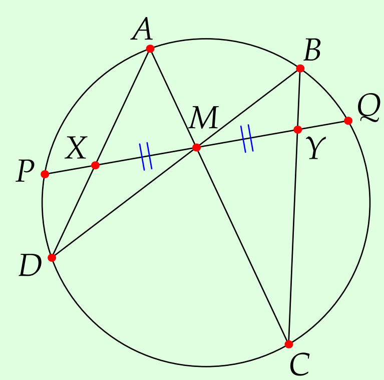
Lemma 2.47. *Let $\overline{AD}$ and $\overline{PQ}$ be chords of a circle which intersect at X. Then*

$$|AX|\,|XD| = |PX|\,|XQ|$$

Theorem 2.48 (Butterfly Theorem). *Given a circle, suppose:*

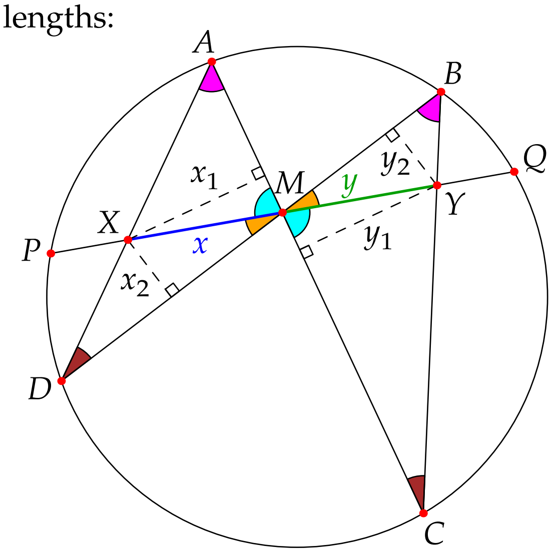
- *$\overline{PQ}$ is a chord with midpoint M.*
- *$\overline{AC}$ and $\overline{BD}$ are chords meeting at M.*
- *$X = \overline{AD} \cap \overline{PQ}$ and $Y = \overline{BC} \cap \overline{PQ}$.*

Then M is the midpoint of $\overline{XY}$.



Proof. For convenience we introduce several numerical lengths:

- $z = |PM| = |MQ|$.
- $x = |XM|$ and $y = |YM|$.
- Drop perpendiculars from X, Y to chords $\overline{AD}, \overline{BC}$, to obtain x_1, x_2, y_1, y_2 as shown.



Our goal is to prove that $x = y$.

First observe that we have two pairs of congruent (vertical) angles at M, and two pairs of congruent inscribed angles at A, B and C, D.

By the AAA theorem, we now have several pairs of similar triangles, whose sides may be compared:

(a) $\frac{x}{x_1} = \frac{y}{y_1}$ and $\frac{x}{x_2} = \frac{y}{y_2}$, from which we see that $\frac{x}{y} = \frac{x_1}{y_1} = \frac{x_2}{y_2}$

(b) $\frac{|AX|}{|BY|} = \frac{x_1}{y_2}$ and $\frac{|XD|}{|YC|} = \frac{x_2}{y_1}$

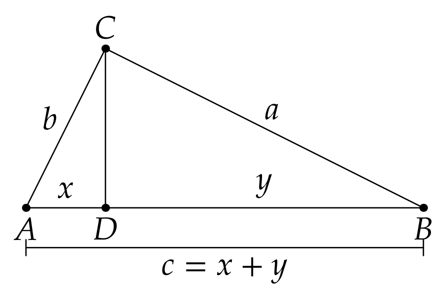
The result follows by combining these observations and applying the Lemma twice:

$$\frac{x^2}{y^2} \overset{(a)}{=} \frac{x_1 x_2}{y_1 y_2} \overset{(b)}{=} \frac{|AX|\,|XD|}{|BY|\,|YC|} \overset{(2.47)}{=} \frac{|PX|\,|XQ|}{|PY|\,|YQ|} = \frac{(z-x)(z+x)}{(z+y)(z-y)} = \frac{z^2 - x^2}{z^2 - y^2}$$
$$\implies x^2(z^2 - y^2) = y^2(z^2 - x^2)$$
$$\implies x^2 = y^2 \implies x = y$$

∎

Exercises 2.5. *Key concepts: Length, angle and area measure, AAA similarity*

1. Which axioms (Hilbert's and angle-measure) prove that $m\angle ABC = m\angle CBA$? Explain.
2. Use similar triangles to prove Lemma 2.47.
3. Let $\triangle ABC$ have a right-angle at C. Drop a perpendicular from C to $\overleftrightarrow{AB}$ at D.
   (a) Prove that D lies between A and B.
   (b) Prove that you have *three* similar triangles
   $$\triangle ACB \sim \triangle ADC \sim \triangle CDB$$
   (c) Use these facts to prove Pythagoras' Theorem.
   (*Use the picture, where a, b, c, x, y are lengths*)



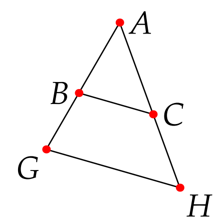
4. In the proof of AAA similarity (Theorem 2.43) explain why

$$d_1 = d_2 \iff \ell \parallel \overline{BC}$$

follows from Playfair's axiom.

(*Hint: compare Exercise 2.1.2 (Thm I. 46) on the construction of a square...*)

5. Prove a simplified version of the *SAS similarity theorem*:



$$\frac{|AB|}{|AG|} = \frac{|AC|}{|AH|} \iff \triangle ABC \sim \triangle AGH$$

(*Hint: construct $\overline{BJ}$ parallel to $\overline{GH}$ and appeal to AAA*)

6. By excluding the other possibilities, prove the converse of the length axiom L4:

If A, B, C are distinct and $|AB| + |BC| = |AC|$, then B lies between A and C.

7. Use Pythagoras' to prove that $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$, that $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$.

8. Prove the converse of Theorem 2.45: if $\sin \angle ABC = \sin \angle A'B'C'$, then $\angle ABC \cong \angle A'B'C'$.

(*Hint: create right-triangles and prove they are similar. Label the side lengths o, a, h, etc.*)

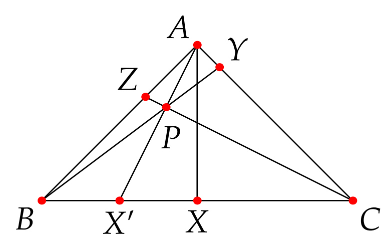
9. We complete the proof of Ceva's theorem.

   (a) If p, q, r, s are non-zero real numbers, verify that $\alpha = \frac{p}{q} = \frac{r}{s} \implies \alpha = \frac{p-r}{q-s}$.

   (b) Assume X, Y, Z satisfy Ceva's formula.

   Define P as the intersection of $\overline{BY}$ and $\overline{CZ}$ and let $\overrightarrow{AP}$ meet $\overline{BC}$ at X'.

   Prove the ($\Rightarrow$) direction of Ceva's theorem by using the ($\Leftarrow$) direction to show that $X' = X$.



10. (a) A *median* of a triangle is a segment from a vertex to the midpoint of the opposite side. Use Ceva's Theorem to prove that the medians of a triangle meet at a point (the *centroid*).

    (b) (Hard) Medians split a triangle into six sub-triangles. Prove that all have the same area.

    (c) Prove that the centroid is exactly $2/3$ of the distance along each median.

11. Prove that similarity of triangles is an equivalence relation.

(*Don't use AAA since its proof requires this fact!*)

12. (Hard) Explain how to prove ($2 \Rightarrow 1$) in the AAA Theorem (2.43).