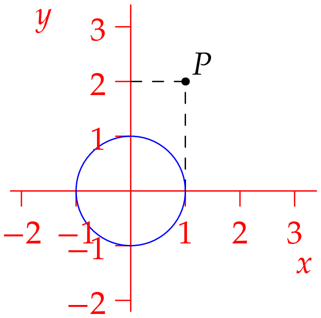
3 Analytic Geometry

Geometry in the style of Euclid and Hilbert is termed *synthetic*: axiomatic, without co-ordinates or explicit numerical measures of angle, length, area or volume. By contrast, the modern practice of geometry is typically *analytic*: reliant on algebra and co-ordinates (including vectors).

Analytic geometry arose in the early 1600s thanks to the work of René Descartes and Pierre de Fermat. The critical development was the *axis*[19] as a fixed reference ruler against which objects can be measured using *co-ordinates*. In honor of Descartes' work, the standard system within analytic geometry is called the Cartesian (literally *of Descartes*) co-ordinate system.

3.1 Cartesian Co-ordinates

Since Cartesian geometry should be very familiar, we merely sketch the core ideas.



- Perpendicular *axes* meet at the *origin* O. The axes are labelled using the real numbers.
- The *co-ordinates* of a point are measured by projecting orthogonally onto the axes. Since co-ordinates are pairs of real numbers, we denote the set of these

  $$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$$

  In the picture, P has co-ordinates $(1,2)$; usually written $P = (1,2)$.
- Points may be manipulated algebraically via *addition* and *scalar multiplication*: if $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then

  $$P + Q := (p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2) \qquad \lambda P := (\lambda p_1, \lambda p_2)$$
- The *length* of a segment is computed using Pythagoras' Theorem

  $$d(P,Q) = |PQ| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

  In the picture, $|OP| = \sqrt{1^2 + 2^2} = \sqrt{5}$. As in Section 2.5, segments are congruent if and only if they have the same length.
- *Curves* are defined using *equations*. For instance, $x^2 + y^2 = 1$ describes the pictured circle; by the distance formula, this is the locus of all points a distance 1 from the origin.

Analytic geometry was originally conceived as a computational toolkit built on top of Euclid's geometry. At first, mathematicians felt the need to justify analytic arguments synthetically lest no-one believe their work.[20] Synthetic geometry is not without its benefits, but its study has become a fringe activity in modern times; co-ordinates are just too useful to ignore!

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[19] Originally there was only one axis, though a second was implied. Very quickly it became standard to have a second axis at right-angles to the first.

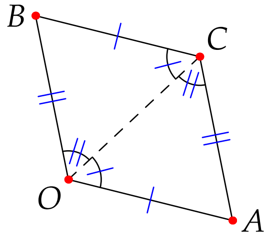
[20] This attitude persisted for some time. For example, the presentation in Issac Newton's groundbreaking *Principia* (1687) was largely synthetic, even though his private derivations made extensive use of co-ordinates and algebra.

It is important to get the history in the right order: in analytic arguments, we may assume anything from Euclid and mix strategies as appropriate. To see this at work, consider a simple result.

Lemma 3.1. *Suppose non-collinear points* $O = (0,0)$, $A = (x,y)$ *and* $B = (v,w)$ *are given and we define* $C := (x+v, y+w)$. *Then* $OACB$ *is a parallelogram.*

Proof. The distance formula shows that opposite sides have the same length

$$|BC| = \sqrt{x^2+y^2} = |OA|, \qquad |AC| = \sqrt{v^2+w^2} = |OB|$$



and therefore congruent.
SSS shows that $\triangle OAC \cong \triangle CBO$.
Euclid's discussion of alternate angles (pages 13–15) forces the opposite sides to be parallel. ■

Lemma 3.1 is essentially vector addition: feel free to use such notation/language if you prefer.

The next result is very useful: it gives an explicit parametrization for all points on a line through two given points.

Lemma 3.2. *The points* X_t *on the line* $\overleftrightarrow{PQ}$ *are in 1–1 correspondence with the real numbers via*

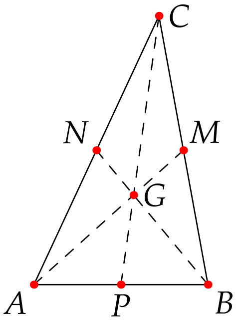
$$X_t = P + t(Q-P) = (1-t)P + tQ$$

Moreover, $d(P, X_t) = |t|\,|PQ|$ *so that* t *measures the (signed) distance along the line.*

The proof is an exercise. As an example of how easy it can be to work in analytic geometry, we apply the Lemma to re-establish a famous result (compare Exercise 2.5.10c where we used Ceva's Theorem).

Theorem 3.3. *The medians of a triangle meet at a point* $2/3$ *of the way along each median.*

Proof. Given $\triangle ABC$, label the midpoints of each side as shown. By Lemma 3.2, these are



$$M = \frac{1}{2}(B+C), \quad N = \frac{1}{2}(A+C), \quad P = \frac{1}{2}(A+B)$$

The point $\frac{2}{3}$ of the way along median $\overline{AM}$ is then

$$G := A + \frac{2}{3}(M-A) = A + \frac{1}{3}(B+C-2A) = \frac{1}{3}(A+B+C)$$

By symmetry (check directly if you like), G is also $\frac{2}{3}$ of the way along the other two medians. ■

The analytic proof relied on adding points as abstract objects, though we could instead have expressed $A, B, C, \ldots$ in co-ordinates. Exercise 2 illustrates exactly this and, in an extension, of the biggest advantages of analytic geometry: the freedom to choose axes and co-ordinates so as to make calculations simple. This trick is essentially Euclid's superposition principle or Hilbert's congruence in disguise: we'll make this correspondence of concepts rigorous in shortly when we discuss *isometries*.

Exercises 3.1. *Key concepts: Analytic geometry is Euclidean geometry plus co-ordinates/algebra*

1. By completing the square, identify the curve described by the equation

$$x^2 + y^2 - 4x + 2y = 10$$

2. (a) Perform a pure co-ordinate proof of Theorem 3.3. For simplicity, arrange the triangle so that $A = (0, 0)$ is the origin and B points along the positive x-axis.

   (b) Descartes and Fermat did not have a fixed perpendicular second axis. Their approach was equivalent to choosing a second axis oriented to make the problem as simple as possible.

   Given $\triangle ABC$, choose axes pointing along $\overline{AB}$ and $\overline{AC}$. Describe the co-ordinates of B and C with respect to such axes. Now give an even simpler proof of the centroid theorem (3.3).

3. Prove Lemma 3.2.

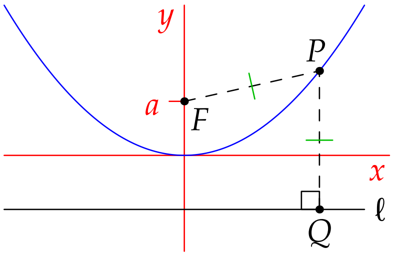
4. A *parabola* is a curve whose points are equidistant from a fixed point F (the *focus*) and a fixed line ℓ (the *directrix*). Otherwise said, if we drop the perpendicular from a point P on the curve to $Q \in \ell$, then the parabola is the locus of all points for which $|FP| = |PQ|$.

   (a) Choose axes so that $F = (0, a)$ and ℓ has equation $y = -a$. Find the equation of the parabola.

   (b) Now let e be any positive constant. Find the equation of the curve if ℓ has equation $y = -\frac{a}{e}$ and we assume $|FP| = e\,|PQ|$.

   How does the type of curve depend on e?

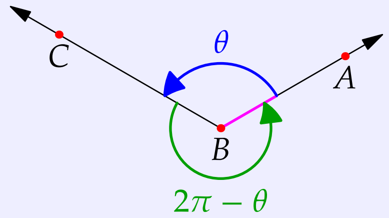
   What happens in the limit $e \to 0^+$?



3.2 Angles and Trigonometry

Angles are defined differently to Section 2.5, though the approach should feel familiar.

Definition 3.4. Suppose A, B, C are distinct points in the plane. Take any circular arc centered at A and define the *radian measure*



$$\measuredangle ABC := \frac{\text{arc-length}}{\text{radius}} \in [0, 2\pi)$$

where arc-length is measured *counter-clockwise* from $\overrightarrow{BA}$ to $\overrightarrow{BC}$.

Radian measure outside $[0, 2\pi)$ are also acceptable, provided they are understood to be reduced modulo 2π: negative angles can therefore be visualized as being measured clockwise.

Since arc-length scales with radius, the definition is independent of the radius of the circular arc. It is important to appreciate the difference between angle measures in our two geometries.

Euclidean geometry Every angle has degree measure strictly between 0° and 180°. Reversed legs produce *congruent* angles (pictorially these are the *same* angle), which always have the *same degree measure*:

$$\angle CBA \cong \angle ABC \quad \text{and} \quad m\angle CBA = m\angle ABC$$

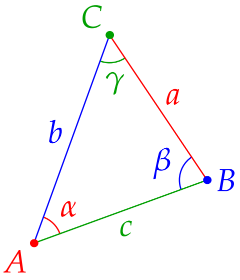
Analytic geometry Reflex angles exist $(> \pi)$. Reversed legs produce *different radian measures*: with regard to the above picture,

$$\measuredangle CBA = 2\pi - \theta \neq \theta = \measuredangle ABC \qquad \text{(unless a straight edge)}$$

However, since one arrangement of legs always has measure $< \pi$,

$$\boxed{\angle XYZ \cong \angle ABC \iff \measuredangle XYZ = \measuredangle ABC \text{ or } \measuredangle CBA \quad (\neq 0, \pi)}$$

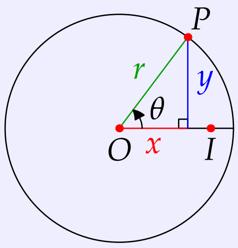
As such, we tend to label angles in a triangle by their measures (degree or radian $< \pi$). Standard convention is shown: $(A, a, \alpha) \leftrightsquigarrow$ (point, length, angle).



Definition 3.5 (Trigonometric Functions). Let O be the origin and I the point $(1, 0)$. Let $P = (x, y)$ lie on a circle of radius r and $\theta = \measuredangle IOP$. We define:

$$\cos\theta := \frac{x}{r} \qquad \sin\theta := \frac{y}{r} \qquad \tan\theta := \frac{y}{x} \quad (x \neq 0)$$

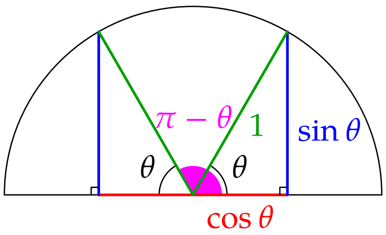
AAA similarity (Thm. 2.43) says these functions are well-defined and independent of r.

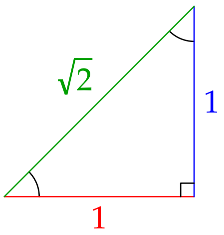


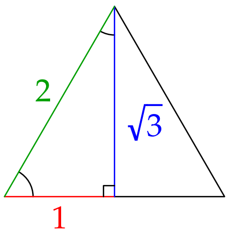
Example 3.6. Basic identities should be clear from the picture, for instance:

$$\cos^2\theta + \sin^2\theta = 1 \text{ (Pythagoras!)} \quad \text{and} \quad \sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

Which well-known facts regarding sine and cosine are illustrated by the following?







Solving Triangles

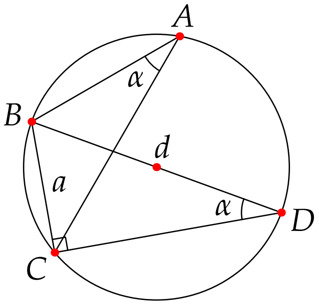
A triangle is described by six values: three side lengths and three angle measures. Euclid's triangle congruence theorems (SAS, ASA, SSS, SAA) say that three of these in suitable combination are enough to recover the rest. In analytic geometry, these calculations typically use the sine and cosine rules.

Theorem 3.7. *Label the sides/angles of $\triangle ABC$ using the standard convention (page 48).*

Sine Rule: *If d is the diameter of the circumcircle, then* $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{d}$

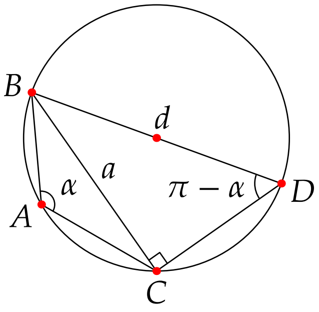
Cosine Rule: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof. We prove the sine rule and leave the cosine rule as an exercise. Everything relies on Corollary 2.33. Draw the circumcircle of $\triangle ABC$. Construct $\triangle BCD$ with diameter $\overline{BD}$; this is right-angled at C by Thales' Theorem. There are two cases:



1. If A and D lie on the same side of $\overleftrightarrow{BC}$, then they share the same arc. But then $\measuredangle BDC = \alpha$ and

$$a = d\sin\measuredangle BDC = d\sin\alpha$$



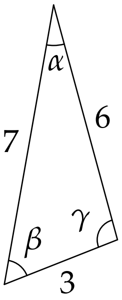
2. If A and D lie on opposite sides of $\overleftrightarrow{BC}$, then the quadrilateral $ABDC$ lies on a circle. Opposite angles at A, D are supplementary, whence

$$\sin\alpha = \sin(\pi - \alpha) = \sin\measuredangle BDC = \frac{a}{d}$$

The two other angle-side combinations follow by permutation. ■

Examples 3.8. We solve two triangles. It is completely acceptable (and equivalent[21]) to so this *without* using the sine or cosine rules: just drop a perpendicular from one vertex to obtain two right triangles.

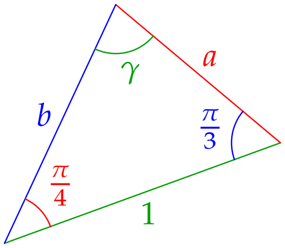
1. The fastest way to solve a triangle given SSS data is to directly compute the three angles using the cosine rule. For instance the given triangle has



$$\alpha = \frac{6^2+7^2-3^2}{2\cdot 6\cdot 7} = \cos^{-1}\frac{19}{21} \approx 25°$$
$$\beta = \frac{3^2+7^2-6^2}{2\cdot 3\cdot 7} = \cos^{-1}\frac{11}{21} \approx 58°$$
$$\gamma = \frac{3^2+6^2-7^2}{2\cdot 3\cdot 6} = \cos^{-1}\frac{-1}{9} \approx 96°$$

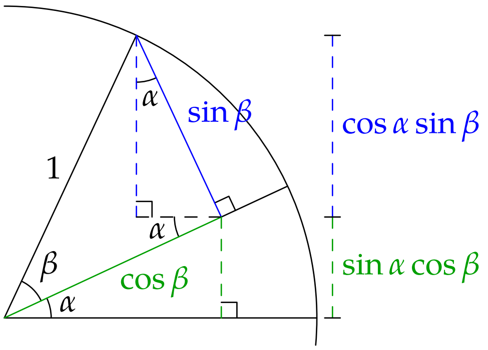
   Once you have α, you could alternatively switch to the sine rule to find β, before finally computing $\gamma = \pi - \alpha - \beta$.

2. To solve the given triangle with ASA data, first find the remaining angle $\gamma = \pi - \frac{\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$ before applying the sine rule



$$\frac{\sin\frac{\pi}{4}}{a} = \frac{\sin\frac{\pi}{3}}{b} = \sin\frac{5\pi}{12} \implies a = \frac{1}{\sqrt{2}\sin\frac{5\pi}{12}} \approx 0.732$$
$$b = \frac{\sqrt{3}}{2\sin\frac{5\pi}{12}} \approx 0.897$$

Multiple-angle formulæ



The picture provides a simple proof of the expressions

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

at least when $\alpha + \beta < \frac{\pi}{2}$. A little algebraic manipulation produces the double-angle and difference formulæ, and verifies that these hold for all possible angle inputs.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

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[21] The proof of the cosine rule and an alternative for the sine rule (Exercise 6) arise this way.

Exercises 3.2. *Key concepts: Radian measure, Congruent angles can have different radian measures, Trigonometric functions, Solving triangles: numerical versions of Euclid's congruence theorems*

1. A triangle has angle of $\frac{2\pi}{3}$ radians between sides of lengths 2 and $\sqrt{3}-1$. Find the length of the remaining side, and the remaining angles.

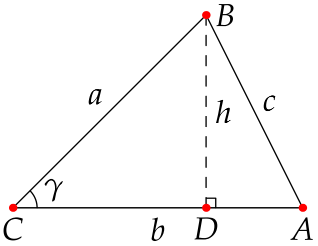
2. Describe the how to solve a triangle given SAA data.

3. Two measurements for the height of a mountain are taken at sea level 5000 ft apart in a line pointing away from the mountain. The angles of elevation to the mountain top from the horizontal are 15° and 13° respectively. What is the height of the mountain?

4. Use a multiple angle formula to find an exact value for $\cos\frac{\pi}{12}$ and thus exact values for the side lengths of the triangle in Example 3.8.2.

5. The area of a triangle is $\frac{1}{2}(\text{base})\cdot(\text{height})$. Repeatedly apply this formula to find an alternative proof of the sine rule without the relationship to the circumcircle.

6. Given $\triangle ABC$, drop a perpendicular from B to $D \in \overleftrightarrow{AC}$ as shown.
   (a) Use Pythagoras' to construct a proof of the cosine rule.
   (b) Is your arguments valid if D is not interior to $\overline{AC}$? Explain.

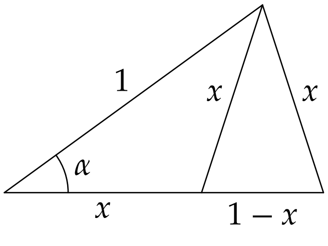


7. The *dot product* of $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is the scalar $A \cdot B := a_1b_1 + a_2b_2$. If $O = (0,0)$ is the origin, apply the cosine rule to $\triangle OAB$ to prove that

$$A \cdot B = |OA|\,|OB| \cos \measuredangle AOB$$

8. Derive the multiple-angle formula for $\sin(\alpha - \beta)$.
   (*Remember that* $0 \le \alpha, \beta, \alpha - \beta < 2\pi$ *so you can't simply switch the sign of* β*!*)

9. Given the arrangement pictured, find x, the radian-measure α and the exact value of $\cos\alpha$.
   (*Hint: first show that you have two similar isosceles triangles*)



10. (a) You are given SSA data for a triangle: sides with lengths $a = 1$ and $b = \sqrt{3}$ and angle $\alpha = \frac{\pi}{6}$. Show that there are *two* triangles satisfying this data.
    (b) (Hard) Suppose you are given (positive) SSA data a, b, α. Show that this data corresponds to at least one real triangle if and only if $0 < \frac{b}{a}\sin\alpha \le 1$. By considering fixed values of b, α and increasing a from zero, explain how the possible number of triangles depends on a, b, α.

3.3 Isometries

At the heart of elementary geometry is *congruence*, the idea that geometric figures that 'line up' when laid on top of each other are essentially identical. In analytic geometry, congruence is described algebraically using *functions*. Everything is motivated by the fact that congruent segments have the same length.

Definition 3.9. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a (*Euclidean*) *isometry* if it preserves lengths:[22]

$$\forall P, Q \in \mathbb{R}^2,\ d\big(f(P), f(Q)\big) = |PQ|$$

Two figures (segments, angles, triangles, etc.) are said to be *isometric* (or *congruent*) precisely when there is an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ mapping one to the other.

Example 3.10. We check that the map $f(x,y) = \frac{1}{5}\big(3x + 4y, 4x - 3y\big) + (3,1)$ is an isometry. If $P = (x,y)$ and $Q = (v,w)$, then

$$\begin{aligned} d\big(f(P), f(Q)\big)^2 &= \left(\frac{3v + 4w - 3x - 4y}{5}\right)^2 + \left(\frac{4v - 3w - 4x + 3y}{5}\right)^2 \\ &= \frac{3^2 + 4^2}{5^2}\big((v-x)^2 + (w-y)^2\big) = |PQ|^2 \end{aligned}$$

Isometric segments are certainly congruent. We should make sure the same holds for angles.

Lemma 3.11. *Isometries preserve (non-oriented) angles: if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then*

$$\angle PQR \cong \angle f(P)f(Q)f(R)$$

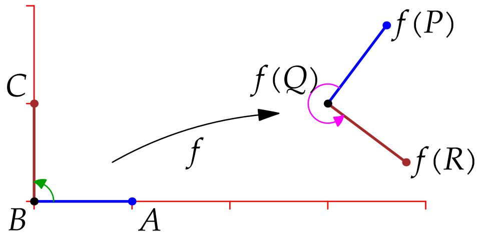
Proof. Since f is an isometry, the sides of $\triangle PQR$ and $\triangle f(P)f(Q)f(R)$ are mutually congruent in pairs. The SSS triangle congruence theorem says that the angles are also mutually congruent. ∎

The Lemma makes clear that isometries are precisely the functions which describe congruence. Examples 3.14 and Exercise 9 expand this idea.

Example (3.10, cont). **Warning**: Isometries can *reverse orientation*! In the picture,

- $\angle ABC \cong \angle f(A)f(B)f(C)$ are both right-angles.
- $\measuredangle ABC = \frac{\pi}{2} \neq \measuredangle f(A)f(B)f(C) = \frac{3\pi}{2}$

The radian measure changed because orientation-reversal flipped counter-clockwise to clockwise!



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[22] In ancient Greek, *iso-metros* is literally *same measure* (*length/distance*).

Classification of Isometries

Our next goal is to confirm our intuition that isometries are rotations, reflections and translations. We start by separating off the translations.

Lemma 3.12. *Suppose f is an isometry. Then $g(X) := f(X) - f(O)$ is an* origin-preserving *isometry. Otherwise said, every isometry f may be written uniquely as a combination*

$$f(X) = g(X) + C$$

where g is an isometry which fixes the origin ($g(O) = O$) and $C = f(O)$ performs a translation.

Proof. Plainly $g(O) = O$. For the rest, just compute:

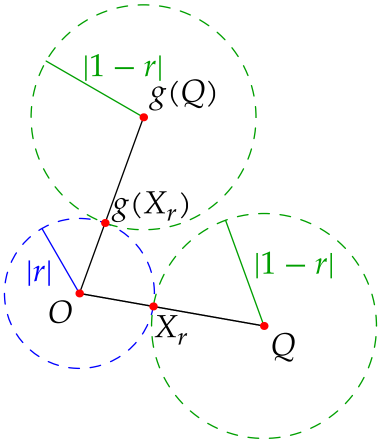
$$g(P) - g(Q) = f(P) - f(Q) \implies d\big(g(P), g(Q)\big) = d\big(f(P), f(Q)\big) = |PQ|$$

■

It thus suffices to describe the origin-preserving isometries g. For these, we make two informal observations that essentially evaluate g in polar co-ordinates..

1. (Scalar Multiplication) Suppose $|OQ| = 1$ and let $X_r = rQ$ for some $r \in \mathbb{R}$. Then,

   - $g(X_r)$ is a distance $|r| = |OX_r|$ from the origin $O = g(O)$.
   - $g(X_r)$ is a distance $|1-r| = |QX_r|$ from $g(Q)$.

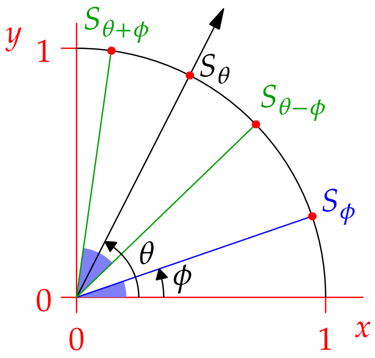
   The point $g(X_r)$ therefore lies on two circles, which necessarily intersect at a single point. We conclude that

   $$g(rQ) = rg(Q)$$

   The picture shows the case $0 < r < 1$, where the uniqueness of intersection follows from $1 = |r| + |1-r|$.

2. The point $g(1,0)$ necessarily lies on the unit circle, and therefore has the form

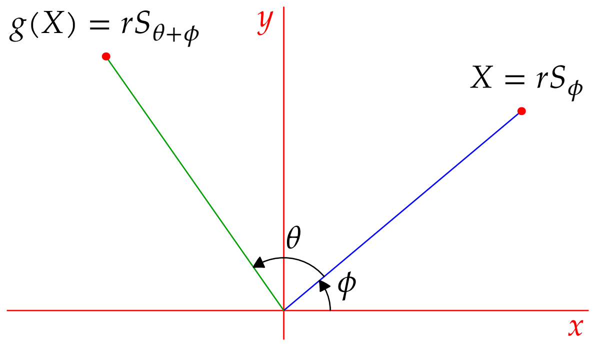
   $$g(1,0) = S_\theta := (\cos\theta, \sin\theta)$$

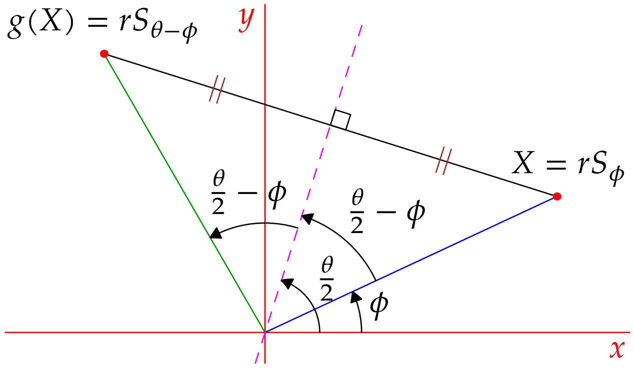
   for some $\theta \in [0, 2\pi)$.
   By preservation of length and angle (Lemma 3.11), any other point $S_\phi = (\cos\phi, \sin\phi)$ on the unit circle must be mapped to the unit circle in such a way that $\measuredangle S_\theta O g(S_\phi) = \pm\phi$.
   Precisely two points satisfy this condition:

   $$g(S_\phi) = S_{\theta\pm\phi} = \big(\cos(\theta\pm\phi), \sin(\theta\pm\phi)\big)$$

Now we put everything together. Express a general point $X = rS_\phi = (r\cos\phi, r\sin\phi)$ in polar co-ordinates and combine the above observations to see that g has one of two forms:



Rotation counter-clockwise by θ



Reflection across the line making angle $\frac{\theta}{2}$ with the positive x-axis

Finally we combine with the Lemma to conclude:

Theorem 3.13. *Every isometry of* $\mathbb{R}^2$ *has the form*

$$f(X) = g(X) + C$$

where g *is a rotation about the origin or a reflection across a line through the origin.*

Calculating with isometries

This benefits enormously from column-vector notation and matrix multiplication. Writing $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} r\cos\phi \\ r\sin\phi \end{smallmatrix}\right)$ for the position vector of $X_r = (x, y) = rS_\phi$ and applying the multiple-angle formulæ, rotation by θ becomes

$$g(\mathbf{x}) = r\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = r\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$$

For reflections, the sign of the second column is reversed: $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$. In vector notation, every isometry therefore has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ where A is an *orthogonal matrix*.[23]

Example (3.10, mk. III). We rewrite the isometry in vector/matrix format:

$$f(\mathbf{x}) = \frac{1}{5}\begin{pmatrix} 3x + 4y \\ 4x - 3y \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Since $\frac{\sin\theta}{\cos\theta} = \frac{4/5}{3/5} = \frac{4}{3}$, we see that the effect of the isometry is first to *reflect* across the line through the origin making angle $\frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.6°$ with the positive x-axis, before *translating* by $(3, 1)$. Compare this description with the original picture!

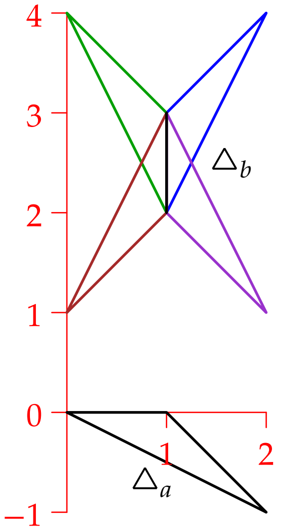
---

[23] An orthogonal matrix satisfies $A^TA = I$. All such have the form $\left(\begin{smallmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \mp b \\ b & \pm a \end{smallmatrix}\right)$ where $a^2 + b^2 = 1$.

Example 3.14. Suppose we have two congruent triangles:

- $\triangle_a$ has vertices $(0,0)$, $(1,0)$ and $(2,-1)$.
- $\triangle_b$ has vertices $(1,2)$, $(1,3)$ and an unknown point P.

We find all isometries transforming $\triangle_a$ to $\triangle_b$ and the location(s) of the third vertex of $\triangle_b$. Let $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ be the isometry. Since $d\big((1,2),(1,3)\big) = 1$ these points must be the images under f of $(0,0)$ and $(1,0)$. There are *four* possibilities:



- If $f(0,0) = (1,2)$ and $f(1,0) = (1,3)$, then $\mathbf{c} = f\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ and
$$A\begin{pmatrix}1\\0\end{pmatrix} + \mathbf{c} = \begin{pmatrix}1\\3\end{pmatrix} \implies A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \implies A = \begin{pmatrix}0 & a_{12}\\ 1 & a_{22}\end{pmatrix}$$
for some a_{12}, a_{22}. Since A is orthogonal, the options are $A = \left(\begin{smallmatrix}0 & \mp1\\ 1 & 0\end{smallmatrix}\right)$ and we obtain two possible isometries:

  $f_1(\mathbf{x}) = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ rotates by 90°, then translates by $\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$.

  $f_2(\mathbf{x}) = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ reflects across $y = x$, then translates.

  The third point of $\triangle_b$ is either $f_1(2,-1) = (2,4)$ or $f_2(2,-1) = (0,4)$.
- $f(0,0) = (1,3)$ and $f(1,0) = (1,2)$ results in two further isometries f_3 and f_4. The details are an exercise.

Isometries and Group Theory

In 1872, Felix Klein suggested that the geometry of a set is the study of its *invariants*: properties preserved by its structure-preserving transformations. In Euclidean geometry, this is the group of *Euclidean isometries* (Exercise 11). Klein's approach provided a method for analyzing and comparing the non-Euclidean geometries beginning to appear in the late 1800s. By the mid 1900s, the resulting theory of *Lie groups* had largely classified classical geometries. Klein's algebraic take on geometry remains dominant in modern mathematics.

Exercises 3.3. *Key concepts: Isometries are congruence as functions, Rotations, Reflections & Translations, Computation using orthogonal matrices*

1. We saw that every isometry (length-preserving map) also preserves angles. Is the opposite true? Does an angle-preserving map necessarily preserve lengths? Explain.
2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry, "reflect across the line through the origin making angle $\frac{\pi}{3}$ with the positive x-axis." Find a 2×2 matrix A such that $f(\mathbf{x}) = A\mathbf{x}$.
3. Describe the geometric effect of the isometry $f(\mathbf{x}) = \frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}\mathbf{x} + \left(\begin{smallmatrix}3\\-2\end{smallmatrix}\right)$
4. Find the remaining isometries f_3 and f_4 in Example 3.14.

5. Find the reflection of the point $(4,1)$ across the line making angle $\frac{1}{2}\tan^{-1}\frac{12}{5} \approx 33.7°$ with the positive x-axis.

   (*Hint: if* $\tan\theta = \frac{12}{5}$, *what are* $\cos\theta$ *and* $\sin\theta$*?*)

6. An origin-preserving isometry $f(\mathbf{v}) = A\mathbf{v}$ moves the point $(7,4)$ to $(-1,8)$.

   (a) If f is a rotation, find the matrix A. Through what angle does it rotate?

   (b) If f is a reflection, find the matrix A. Across which line does it reflect?

7. Let $ABCD$ be the rectangle with vertices $A = (0,0)$, $B = (4,0)$, $C = (4,3)$, $D = (0,3)$. Suppose an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps $ABCD$ to a new rectangle $PQRS$ where

$$P = f(A) := (2,4) \quad \text{and} \quad R = f(C) := (2,9)$$

   Find all possible isometries f and the remaining points $Q = f(B)$ and $S = f(D)$.

8. (a) If $A = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ and $\mathbf{p}$ is constant, explain why $f(\mathbf{x}) = A(\mathbf{x}-\mathbf{p})+\mathbf{p} = A\mathbf{x}+(I-A)\mathbf{p}$ rotates by θ around the point with position vector $\mathbf{p}$.

   (b) Suppose $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ rotates the plane around the point $P = (-2,1)$ by an angle $\theta = \tan^{-1}\frac{3}{4}$. Find A and $\mathbf{c}$.

   (c) Suppose f rotates by θ around $\mathbf{p}$ and g rotates by ϕ around $\mathbf{q}$ where θ, ϕ are non-zero.

      i. If $\theta + \phi \neq 2\pi$, show that $f \circ g$ is a rotation: by what angle and about which point?

      ii. What happens instead if $\theta + \phi = 2\pi$?

9. (Hard) Suppose $\triangle ABC \cong \triangle PQR$. Prove that there exists an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ mapping one triangle to the other. How many distinct such isometries could there be, and how does this number depend on the triangles?

10. Make a circle-intersection argument (page 53) to prove that for any isometry f,

$$f\big((1-t)P + tQ\big) = (1-t)f(P) + tf(Q)$$

11. Throughout this question, we use the notation $f_{A,\mathbf{c}} : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{c}$.

   (a) Prove that isometries obey the composition law $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{AB,\mathbf{c}+A\mathbf{d}}$.

   (b) Find the inverse function of the isometry $f_{A,\mathbf{c}}$. Otherwise said, if $f_{A,\mathbf{c}} \circ f_{C,\mathbf{d}} = f_{I,\mathbf{0}}$, where I is the identity matrix, how do $B, \mathbf{d}$ depend on $A, \mathbf{c}$?

   (c) Verify that the following composition $f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f_{A,\mathbf{c}}^{-1}$ is a translation.

   Part (a) can be written using augmented matrices: $(A\,|\,\mathbf{c})(B\,|\,\mathbf{d}) := (AB\,|\,\mathbf{c} + A\mathbf{d})$.
   If you know group theory, parts (a) and (b) are the closure and inverse properties for the group of Euclidean isometries E. Part (c) says that the translations T form a normal subgroup; E is therefore a semi-direct product of T and the orthogonal group of origin-preserving isometries $E = T \rtimes O_2(\mathbb{R})$.

3.4 Birkhoff's Axiomatic System for Analytic Geometry (non-examinable)

Recall that analytic geometry was originally conceived as a bolt-on to Euclidean geometry. In 1932, George David Birkhoff provided an axiomatization of analytic geometry in its own right.

Background Assume the usual properties/axioms of the real numbers as a complete ordered field. Birkhoff's approach is typical of modern axiomatic systems in that it is built on top of pre-existing systems (set theory, complete ordered fields, etc.).

Undefined terms Two objects: *Point* & *line*.
Two functions: *distance* d & *angle measure* $\measuredangle$. If the set of points is $\mathcal{S}$, then,

$$d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_0^+, \qquad \measuredangle : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0, 2\pi)$$

Axioms Birkhoff has four axioms (in addition to those required for the real numbers, etc.).

Euclidean Given two distinct points, there exists a unique line containing them.]

Ruler Points on a line ℓ are in bijective correspondence with the real numbers in such a way that if t_A, t_B correspond to $A, B \in \ell$, then $|t_A - t_B| = d(A, B)$.

Protractor The rays emanating from a point O are in bijective correspondence with the set $[0, 2\pi)$, so that if α and β correspond to rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then

$$\measuredangle AOB \equiv \beta - \alpha \pmod{2\pi}$$

This correspondence is continuous in A and B.

SAS similarity[24] If $\triangle ABC$ and $\triangle XYZ$ satisfy

$$\measuredangle ABC = \measuredangle XYZ, \quad \text{and} \quad \frac{d(A,B)}{d(X,Y)} = \frac{d(B,C)}{d(Y,Z)}$$

then the remaining angles have equal measure and the final sides are in the same ratio (i.e., $\triangle ABC \sim \triangle XYZ$).

Definitions As with Hilbert, some of these are required before later axioms make sense. For instance, the definition of *ray* comes before the *protractor* axiom.

Betweenness B lies between A and C if $d(A, B) + d(B, C) = d(A, C)$

Segment $\overline{AB}$ consists of the points A, B and all those between

Ray $\overrightarrow{AB}$ consists of the segment $\overline{AB}$ and all points C such that B lies between A and C.

Basic shapes Triangles, circles, etc.

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[24] As with Hilbert, Birkhoff makes SAS an *axiom*, though Birkhoff's version is stronger in that it also applies to similar triangles. SAS *congruence* is the special case where the similarity ratio is 1.

Analytic Geometry as a Model

The axioms should feel familiar. Being shorter than Hilbert's list, and being built on familiar notions such as the real line, the axioms are somewhat easier to understand and to visualize. There is something to *prove* however; indeed the major point of Birkhoff's system!

Theorem 3.15. *Cartesian analytic geometry is a model of Birkhoff's axioms.*

Recall what this requires: we must provide a *definition* of each of the undefined terms and prove that these satisfy each of Birkhoff's axioms. Here are suitable definitions for Cartesian analytic geometry:

Point An ordered pair (x,y) of real numbers.

Distance $d(A,B)=\sqrt{(A_x-B_x)^2+(A_y-B_y)^2}$

Line All points satisfying a linear equation $ax+by+c=0$.

Angle Define column vectors as differences ($\mathbf{v}=P-O$ and $\mathbf{w}=Q-O$) and consider the matrix $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Now define angle via

$$\cos\measuredangle POQ=\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\quad\text{where}\quad\measuredangle POQ\in\begin{cases}[0,\pi] & \iff \mathbf{w}\cdot J\mathbf{v}\geq 0\\ (\pi,2\pi) & \iff \mathbf{w}\cdot J\mathbf{v}<0\end{cases}\tag{$*$}$$

In essence J is rotate counter-clockwise by $\frac{\pi}{2}$. Cosine may be defined using power series, so no pre-existing geometric meaning is necessary.

Proof. (Euclidean axiom) If (x_1,y_1) and (x_2,y_2) satisfy $ax+by+c=0$ then

$$a(x_1-x_2)+b(y_1-y_2)=0$$

whence $a=y_1-y_2$, $b=x_2-x_1$ up to scaling. It follows that the line has equation

$$(y_1-y_2)x+(x_2-x_1)y+x_1y_2-x_2y_1=0$$

unique up to multiplication of all three of a,b,c by a non-zero constant.

The remaining axioms are exercises. ∎

Exercises 3.4. *Key concepts: Analytic geometry as an axiomatic system*

1. Prove that the ruler axiom is satisfied:

   (a) First show that if $P\neq Q$ lie on ℓ, then any point A on the line has the form

   $$A=P+\frac{t_A}{d(P,Q)}(Q-P)\qquad\text{where } t_A\in\mathbb{R}$$

   (b) Use this formula to verify that $d(A,B)^2=(t_A-t_B)^2$.

2. Let $\mathbf{i} = \binom{1}{0}$. Given any non-zero point B, define $\mathbf{b} = B - O$ and let $\beta = \cos^{-1} \frac{\mathbf{i} \cdot \mathbf{b}}{|\mathbf{b}|}$ in accordance with $(*)$. This is a continuous function of $\mathbf{b}$.
   (a) If $\hat{B}$ is any other point on the same ray $\overrightarrow{OB}$, explain why we get the same value β. (β *is thus a continuous function of* B)
   (b) If $B = (x, y)$, what are values $\cos \beta$ and $\sin \beta$?
   (c) Suppose A corresponds to α under this identification. Evaluate $\cos(\beta - \alpha)$ and therefore prove that the protractor axiom is satisfied.

3. Use the cosine rule (Theorem 3.7) to prove that the SAS similarity axiom is satisfied.

4. (Hard) Can you see where uniqueness of parallels comes from in Birkhoff's system? Can you give a proof of this fact, or that the sum of the angles in a triangle is π radians?

3.5 The Complex Plane

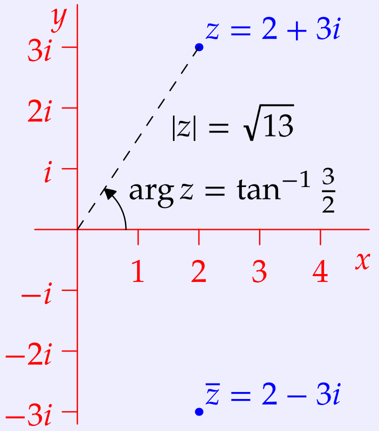
Complex numbers date back at least to 16th century Italy, though their application to geometry really begins with Leonhard Euler (1707–83) who identified the set of complex numbers $\mathbb{C}$ with the Cartesian plane in what is now known as the *Argand diagram*.

Definition 3.16. Let i be an abstract symbol satisfying the property $i^2 = -1$.

Given real numbers x, y, the *complex number* $z = x + iy$ is simply the point (x, y) in the standard Cartesian plane. In the language of linear algebra, $\mathbb{C}$ is a vector space over $\mathbb{R}$ with basis $\{1, i\}$.

Given a complex number $z = x + iy$, its:

- *Complex conjugate* $\overline{z} = x - iy$ is its *reflection* across the real axis.
- *Modulus* $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ is its distance from the origin.
- *Argument* $\arg z$ is the radian measure (counter-clockwise) of the angle between the positive real axis and the ray $\overrightarrow{0z}$.



The algebra of the complex numbers screams *geometry*! Definition 3.16 already describe length & angle-measure and reflection in the real axis. Two other aspects of basic geometry are immediate:

Addition For any fixed $w \in \mathbb{C}$ the map $z \mapsto z + w$ *translates* all points by w. If this feels too quick, write everything in real and imaginary parts. If $z = x + iy$ and $w = p + iq$ are given, then

$$x + iy \mapsto (x + p) + i(y + q) \qquad \text{corresponds to} \qquad (x, y) \mapsto (x + p, y + q)$$

Scalar multiplication If $\lambda \in \mathbb{R}$, then the map $z \mapsto \lambda z$ *scales* distances from the origin. This algebraically encodes *similarity*.

Complex Multiplication & Rotation about the origin

Complex multiplication follows the standard algebraic rules while using $i^2 = -1$ to simplify.

Example 3.17. A simple example of multiplication of complex numbers:

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2 \cdot 4 + 2 \cdot 5i + 3i \cdot 4 + 3i \cdot 5i && \text{(distributivity)}\\ &= 8 + 10i + 12i - 15 && \text{(use } i^2 = -1 \text{ to simplify)}\\ &= -7 + 22i\end{aligned}$$

What we really care about is the geometric interpretation of multiplication. To start visualizing this, consider multiplication by i,

$$iz = i(x + iy) = -y + ix \qquad \text{corresponds to} \qquad (x, y) \mapsto (-y, x)$$

This is the result of *rotating* z counter-clockwise $\frac{\pi}{2} = 90°$ radians about the origin.

To obtain all rotations and reflections, we need an alternative description of a complex number.

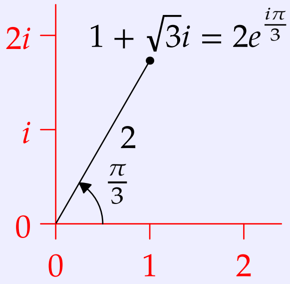
Lemma 3.18. *Suppose* $\theta, \phi \in \mathbb{R}$ *and* $n \in \mathbb{Z}$.

1. (*Euler's Formula*) *For any* $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$.
2. (*Exponential laws*) $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$ *and* $(e^{i\theta})^n = e^{in\theta}$ *for any* $n \in \mathbb{Z}$.

Arguments for both parts are an exercise. Part 1 can be treated a a definition. Evaluating at $\theta = \pi$ yields the famous *Euler identity* $e^{i\pi} = -1$.

Definition 3.19 (Polar form of a complex number). Let $z = x + iy$ be a non-zero complex number with modulus $r = |z|$ and argument $\theta = \arg z$. Using polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ and Euler's formula, we obtain the *polar form*

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$



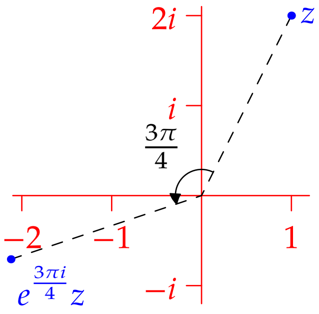
Now consider the effect of multiplying a complex number $z = re^{i\phi}$ by $e^{i\theta} = \cos\theta + i\sin\theta$. By the Lemma

$$e^{i\theta}z = re^{i\theta}e^{i\phi} = re^{i(\theta+\phi)}$$

has the same modulus (r) as z but a new argument...

Theorem 3.20. *For any* $\theta \in \mathbb{R}$ *and* $z \in \mathbb{C}$, *the complex number* $e^{i\theta}z$ *is obtained by rotating* z *counter-clockwise about the origin through an angle* θ.

Example 3.21. To rotate $z = 1 + 2i$ counter-clockwise by $\frac{3\pi}{4}$ radians, we must multiply by



$$e^{\frac{3\pi i}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}(-1+i)$$

That is,

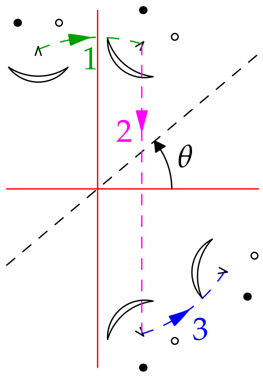
$$e^{\frac{3\pi i}{4}}z = \frac{1}{\sqrt{2}}(-1+i)(1+2i) = -\frac{1}{\sqrt{2}}(3+i)$$

It is acceptable to convert to (or keep everything) in polar form. In this case, however, the result isn't pretty:

$$z = \sqrt{5}e^{i\tan^{-1}2} \implies e^{\frac{3\pi i}{4}}z = \sqrt{5}e^{\frac{3\pi i}{4} + i\tan^{-1}2}$$

General Rotations and Reflections

Recall (Definition 3.16) how complex conjugation reflects across the horizontal axis. General reflections may be described by combining complex conjugation and rotations. For instance, to reflect across the line making angle θ with the positive real axis, we rotate the plane so that the reflection appears vertical:[25]



1. Rotate the plane *clockwise* by θ, that is $z \mapsto e^{-i\theta}z$.
2. Reflect across the real axis via complex conjugation.
3. Rotate counter-clockwise by θ.

Combining these steps proves the main result.

Theorem 3.22. *To reflect z across the line making angle θ with the positive real axis, we compute*

$$z \mapsto e^{i\theta}(\overline{e^{-i\theta}z}) = e^{2i\theta}\overline{z}$$

Example 3.23. We reflect $z = -2 + 3i$ across the line through the origin and $w = \sqrt{3} + i$. First compute $\theta = \arg w = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. The desired point is therefore

$$e^{2i\theta}z = e^{\frac{i\pi}{3}}(-2-3i) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(-2-3i) = \left(\frac{3\sqrt{3}}{2} - 1\right) - \left(\sqrt{3} + \frac{3}{2}\right)i$$

To describe general rotations and reflections about arbitrary points/lines, we combine our approach with *translations* (compare Exercise 3.3.8).

Corollary 3.24. *Let $\theta \in \mathbb{R}$ and $w \in \mathbb{C}$ be given.*

1. *To rotate by θ about w, compute $z \mapsto e^{i\theta}(z - w) + w$.*
2. *To reflect across the line making angle θ through w, compute $z \mapsto e^{2i\theta}(\overline{z} - \overline{w}) + w$.*

Example 3.25. The combination of translation by $-i$, rotation by $\frac{\pi}{3}$ around the origin, then translation by 1, may be expressed

$$z \mapsto e^{\frac{\pi}{3}}\big(z - i\big) + 1 = i + e^{\frac{\pi}{3}}\big(z - i\big) + 1 - i$$

Alternatively, this is rotation by $\frac{\pi}{3}$ around i followed by translation by $1 - i$.

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[25] This approach (and both upcoming results) is a version of *algebraic conjugation* (not to be confused with complex conjugation!). If f is a function, then its conjugate by an invertible function g is the composition $g \circ f \circ g^{-1}$. For instance rotation by θ around w is the conjugation of rotation about the origin ($f(z) = e^{i\theta}$) by a translation ($g(z) = z + w$). Very loosely, any conjugate of f does the same thing as f relative to a new frame of reference.

A Dictionary of Basic Geometric Transformations

We have now described all Euclidean isometries (Section 3.3) using complex numbers. Here is the full dictionary for comparing the two appraoches.[26]

Isometry/Transformation	Complex numbers	Matrices/vectors
Addition/Translation	$z+w=(x+iy)+(u+iv)$	$\mathbf{z}+\mathbf{w}=\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}u\\v\end{pmatrix}$
Scaling	$\lambda z=(\lambda x)+i(\lambda y)$	$\lambda\mathbf{z}=\begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}$
Rotation CCW by $\frac{\pi}{2}$	$z\mapsto iz$	$\mathbf{z}\mapsto\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathbf{z}$
Rotation CCW by θ	$z\mapsto e^{i\theta}z$	$\mathbf{z}\mapsto\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\mathbf{z}$
Vertical reflection	$z\mapsto\overline{z}$	$\mathbf{z}\mapsto\begin{pmatrix}1&0\\0&-1\end{pmatrix}\mathbf{z}$
Reflection across line with slope $\frac{\theta}{2}$	$z\mapsto e^{i\theta}\overline{z}$	$\mathbf{z}\mapsto\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}\mathbf{z}$

It might be surprising to learn that the application of complex numbers to geometry came before vectors and matrices! During the 1800s, mathematicians tried unsuccessfully to replicate the complex number approach in higher dimensions. Their failure ultimately led, via Hamilton's *quaternions*, to the widespread adoption of vectors and linear algebra/matrix calculations.

While matrices have a big advantage in being applicable in arbitrary dimensions, there are good reasons still to employ complex numbers for planar geometry. With a little practice they are quicker to use (less writing). Complex algebra also admits further (non-isometric) transformations: for instance $z\mapsto\overline{z}^{-1}$ is *reflection in a circle*. We'll discuss some of this at the end of Chapter 4.

Exercises 3.5. *Key concepts: Representing points as complex numbers, Polar form, Representing isometries using complex numbers*

1. Use complex numbers to compute the result of the following transformations. Answer in either standard or polar form.
   (a) Rotate $3-5i$ counter-clockwise around the origin by $\frac{3\pi}{4}$ radians.
   (b) Reflect $2-i$ across the line joining $1+i\sqrt{3}$ and the origin.
   (c) Reflect $1+i$ across the line through the origin making angle $\frac{\pi}{5}$ radians with the positive real axis.
2. Find the reflection of the point $(2,3)$ across the line through the origin making angle $\frac{3\pi}{8}$ with the positive x-axis. Give your answer using both complex numbers and matrices/vectors.
3. Repeat the previous question for the point $(3,4)$ and the angle $\frac{5\pi}{12}=75°$.

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[26] Scaling isn't an isometry, but it is worth including in the dictionary nonetheless.

4. Describe the geometric effect of the map $z \mapsto \frac{1}{\sqrt{2}}(-1-i)\big(\overline{z}-3+4i\big)$.

   (*Hint: compare Example 3.25*)

5. (Hard) Consider the line ℓ through the origin and $\left(\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}\right)$. Compute the result of reflecting $-2+3i$ across ℓ.

6. By letting $n = 3$ in Lemma 3.18, prove that

   $$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

   Find a corresponding trigonometric identity for $\sin 3\theta$.

7. We prove Lemma 3.18.

   (a) Prove part 1 using what you learned about Maclaurin series in calculus.
   (You can alternatively prove this using the existence & uniqueness theorem from elementary differential equations.)

   (b) Prove part 2 using the trigonometric multiple angle formulæ.

8. (Hard) Recall the comment on page 63 about how the function $f(z) = \overline{z}^{-1}$ describes *reflection in the circle*.

   Describe the output curve if all points on the circle $\left|z - \frac{1}{2}\right| = \frac{1}{4}$ are fed to this function.

   Moreover, for any input $z \neq 0, 1$, show that the output $f(z)$ lies outside the unit circle on the ray $\overrightarrow{0z}$.