

4 Hyperbolic Geometry

4.1 History: Saccheri, Lambert and Absolute Geometry

For 2000 years after Euclid, many mathematicians believed that his parallel postulate (page 12) could not be an independent axiom. Much effort was expended trying to demonstrate this, not least by Giovanni Saccheri (1667–1733) & Johann Lambert (1728–1777); both attempted to force contradictions by assuming the negation of the parallel postulate. While this approach ultimately failed, their insights provided the foundation of a new *non-Euclidean* geometry.

Absolute and Non-Euclidean Geometry

Before considering the work of Saccheri and Lambert, we define some terms and recall our earlier discussion of parallels (pages 13–17).

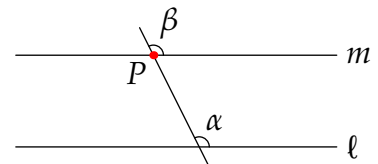
Definition 4.1. *Absolute* or *neutral* geometry is the axiomatic system comprising all of Hilbert’s axioms except Playfair (page 21). Euclidean geometry is a special case of absolute geometry. A *non-Euclidean geometry* is (loosely) a model satisfying most of Hilbert’s axioms but for which parallels might not exist or be unique:

There exists a line ℓ and a point P not on ℓ which lies on *no parallel* or *at least two parallel lines* to ℓ .

Spherical geometry (page 16) is non-Euclidean since there are no parallel lines: Hilbert’s axioms I-2 and O-3 are false, as is the exterior angle theorem.

Results in absolute geometry The conclusions of Euclid’s first 28 theorems are valid.

- Basic constructions: midpoint, bisectors, perpendiculars, etc.
- Triangle congruence theorems: SAS, ASA, SAA, SSS.
- Exterior angle theorem and its consequences:



- Side-angle comparison and the triangle inequality (Exercise 2.3.6).
- *Existence* of a parallel m to a line ℓ through a point $P \notin \ell$: as pictured $\alpha \cong \beta \implies \ell \parallel m$.

Arguments making use of unique parallels Since the following were proved using Playfair’s axiom or the parallel postulate, their arguments are false in absolute geometry:

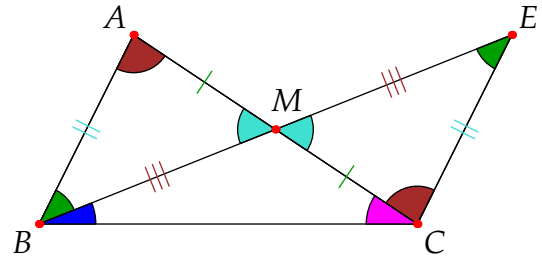
- A line crossing parallel lines makes congruent angles: in the picture, $\ell \parallel m \implies \alpha \cong \beta$. This is the *uniqueness* claim in Playfair: the parallel m to ℓ through P is unique.
- Angles in a triangle sum to a straight-edge (180°).
- Constructions of squares/rectangles.
- Pythagoras’ Theorem.

While our *arguments* for the above are false in absolute geometry, we cannot immediately claim that the *results* are also; there might be alternative proofs! To see that these results truly require unique parallels, we must exhibit a *model* of absolute geometry in which they are false.²⁷ Developing such a model is the primary purpose of this chapter. Its existence explains why Saccheri and Lambert ultimate goal was unrealized: the parallel postulate (Playfair) is indeed independent of Euclid’s (Hilbert’s) other axioms.

The Saccheri–Legendre Theorem

We turn now to Saccheri’s consideration of triangles, suitably modernized. We work in absolute geometry, starting with an extension of Euclid’s proof of the exterior angle theorem.

Suppose $\triangle ABC$ has angle-sum Σ_{\triangle} , let $\alpha_0 = m\angle ABC$ and construct M and E following Euclid. Observe:



1. $\angle ACB + \angle CAB \cong \angle ACB + \angle ACE < 180^\circ$ is the exterior angle theorem. More generally, the exterior angle theorem guarantees that the *sum of any two angles in a triangle is strictly less than 180°* .
2. $\triangle ABC$ and $\triangle EBC$ have the *same angle-sum*—just look at the picture!

$$\Sigma_{\triangle} = \text{red} + \text{green} + \text{blue} + \text{pink}$$

Remember that we do not know whether $\Sigma_{\triangle} = 180^\circ$!

3. $\triangle EBC$ has at least one angle ($\angle EBC$ or $\angle BEC$) measuring at most $\frac{1}{2}\alpha_0 = \frac{1}{2}m\angle ABC$.

Iterate this construction: if $\angle EBC$ is smaller, start by bisecting \overline{CE} ; otherwise bisect \overline{BC} ...

The result is an infinite sequence of triangles $\Delta_1 = \triangle EBC, \Delta_2, \Delta_3, \dots$ with two properties:

- (a) All triangles in the sequence have *same angle-sum* $\Sigma_{\Delta} = \Sigma_{\Delta_1} = \Sigma_{\Delta_2} = \dots$.
- (b) The n^{th} triangle Δ_n has at least one angle measuring $\alpha_n \leq \frac{1}{2^n}\alpha_0$.

Now suppose, for contradiction, that $\Sigma_{\Delta} = 180^\circ + \epsilon$ is *strictly greater* than 180° . Since $\lim \frac{1}{2^n} = 0$, we may choose n large enough to guarantee $\alpha_n < \epsilon$:

$$n > \log_2 \frac{\alpha_0}{\epsilon} \implies \alpha_n < \epsilon$$

But then the sum of the *other two* angles in Δ_n would be *greater than 180°* , contradicting the exterior angle theorem (observation 1)! We have proved a famous result.

Theorem 4.2 (Saccheri–Legendre). *Triangles in absolute geometry have angle-sum $\Sigma_{\Delta} \leq 180^\circ$.*

Saccheri’s failed hope was to prove *equality* without invoking the parallel postulate.

²⁷The model must be of *absolute* geometry. For instance, the angles in a spherical triangle sum to more than 180° , but we can’t guarantee that the reason for this is a *only* the lack of parallels, since spherical geometry fails two of Hilbert’s other axioms.

Saccheri and Lambert Quadrilaterals

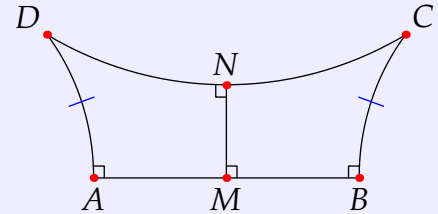
Two families of quadrilaterals in absolute geometry are named in honor of these pioneers.

Definition 4.3. In absolute geometry, a *Lambert quadrilateral* $AMND$ has three right-angles.

A *Saccheri quadrilateral* $ABCD$ satisfies

$$\overline{AD} \cong \overline{BC} \quad \text{and} \quad \sphericalangle DAB = \sphericalangle CBA = 90^\circ$$

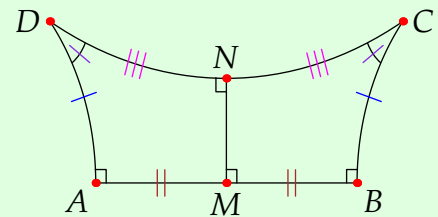
\overline{AB} is the *base* and \overline{CD} the *summit*. The interior angles at C and D are the *summit angles*.



We draw these with curved sides to indicate that the summit angles need not be right-angles, though we haven't yet exhibited a model which shows they could be anything else. Regardless of how they are drawn, \overline{AD} , \overline{BC} and \overline{CD} are all *segments*!

Lemma 4.4. *Saccheri and Lambert quadrilaterals exhibit basic symmetry as pictured.*

1. If the base and summit of a Saccheri quadrilateral are bisected, we obtain two congruent Lambert quadrilaterals.
2. The summit angles of a Saccheri quadrilateral are congruent.
3. In **Euclidean geometry**, Saccheri and Lambert quadrilaterals are **rectangles**: the summit angles are also right-angles, and the summit and base are congruent.



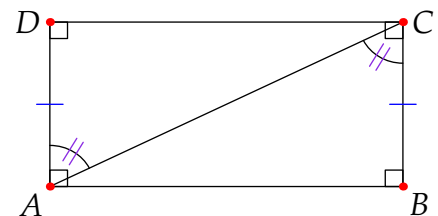
We could interpret part 3 as saying that Saccheri and Lambert quadrilaterals are as close as we can get to rectangles in absolute geometry. The Lemma is mostly a recapitulation of Euclid's Book I construction of a square (Thm. I. 46, Exercise 2.1.2). We leave parts 1 and 2 as exercises: remember that parts 1 and 2 must be proved without reference to Playfair's Axiom!

Proof of part 3. Given a Lambert quadrilateral, a second congruent copy forms a Saccheri quadrilateral: if this last is a rectangle, so is the former. It suffices therefore to prove the result for Saccheri quadrilaterals.

Suppose $ABCD$ is a Saccheri quadrilateral with base \overline{AB} . Following the exterior angle theorem, since \overleftrightarrow{AB} is a crossing line making congruent right-angles we conclude that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

The extended summit \overleftrightarrow{CD} also crosses the same parallels. By the parallel postulate, the summit angles sum to a straight-edge. Since these are congruent (part 2), both are right-angles.

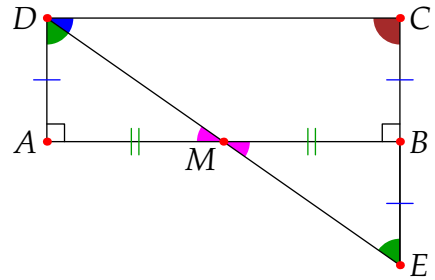
Finally, join \overline{AC} . Again by the parallel postulate, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ says the alternate angles are congruent ($\sphericalangle DAC \cong \sphericalangle BCA$). By SAS the triangles are congruent ($\triangle DAC \cong \triangle BCA$); in particular so also are the summit and base ($\overline{AB} \cong \overline{CD}$) of the original quadrilateral.



We now show that drawing a Saccheri quadrilateral with *non-obtuse* summit angles is justified.

Theorem 4.5. *In absolute geometry, the summit angles of a Saccheri quadrilateral measure $\leq 90^\circ$.*

Proof. Suppose $ABCD$ is a Saccheri quadrilateral with base \overline{AB} . Extend \overline{CB} to E , on the opposite side of \overline{AB} to C , such that $\overline{BE} \cong \overline{DA}$.



Let M be the midpoint of \overline{AB} and join to D and E .

SAS implies $\angle DAM \cong \angle EBM$. In particular, the **vertical angles** at M are congruent, whence M lies on \overline{DE} .

By the Saccheri–Legendre Theorem, the (congruent) **summit angles** at C and D sum to

$$\begin{aligned} \angle ADC + \angle DCB &= \angle ADM + \angle EDC + \angle DCE \\ &= \angle CED + \angle EDC + \angle DCE \\ &\leq 180^\circ \end{aligned}$$

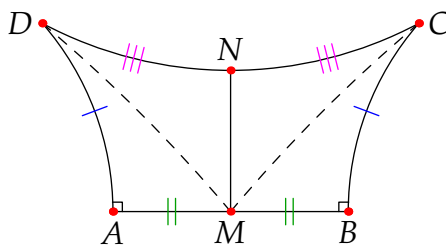
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The next step is to see if there exists a model of absolute geometry in which the summit angles are *strictly less* than a right-angle...

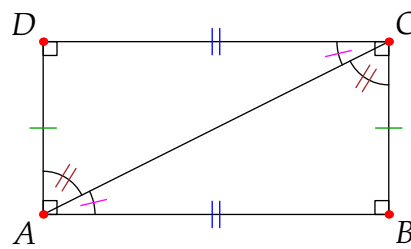
Exercises 4.1. *Key concepts: Absolute Geometry, $\Sigma_\Delta \leq 180^\circ$, Saccheri & Lambert quadrilaterals*

Work in absolute geometry; you cannot use Playfair’s Axiom or the parallel postulate!

1. Use the first picture below to prove parts 1 and 2 of Lemma 4.4.
2. Use the first picture to give an alternative proof of Theorem 4.5.



Exercises 1 & 2



Exercise 3

3. Suppose $ABCD$ has four right-angles. Use the Saccheri-Legendre Theorem to prove that \overline{AC} splits $ABCD$ into two congruent triangles; conclude that $ABCD$ is a rectangle.
(This is easy in Euclidean geometry, but shows that we needn’t have invoked the parallel postulate for a second time in Lemma 4.4, part 3.)
4. Suppose a pair of Saccheri quadrilaterals have congruent bases and perpendicular sides. Prove that the quadrilaterals are congruent.
5. (Hard) Suppose two Saccheri quadrilaterals have congruent summits and congruent perpendicular sides. Prove that the quadrilaterals are congruent.

4.2 Models of Hyperbolic Geometry

In the 1820-30s, János Bolyai, Carl Friedrich Gauss and Nikolai Lobachevsky independently took the next step.²⁸ Rather than attempting to establish the parallel postulate as a theorem within Euclidean geometry, they defined a new geometry based on an alternative axiom of parallels.

Axiom 4.6 (Bolyai–Lobachevsky/Hyperbolic Postulate). Given a line ℓ and a point $P \notin \ell$, there exist *at least two* parallel lines to ℓ through P .

Hyperbolic Geometry is the resulting axiomatic system: Hilbert with Playfair’s axiom replaced by the hyperbolic postulate. Consistency was proved in the late 1800s by Beltrami, Klein and Poincaré, each of whom created models by defining *point*, *line*, etc., in novel ways.

The Poincaré Disk Model

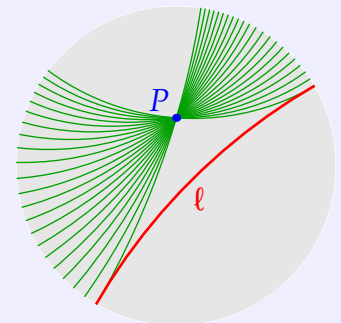
One of the simplest models is named for Poincaré, though it was first proposed by Beltrami.²⁹

Definition 4.7. The *Poincaré disk* is the interior of the unit circle

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \quad \text{or} \quad \{z \in \mathbb{C} : |z| < 1\}$$

A *hyperbolic line* is either a diameter or a circular arc meeting the unit circle at right-angles.

The picture shows a **hyperbolic line** ℓ and a **point** P : also drawn are several **parallel hyperbolic lines** to ℓ passing through P .



Points on the boundary circle are termed *omega-points*: these are *not* in the Poincaré disk and are essentially ‘points at infinity.’

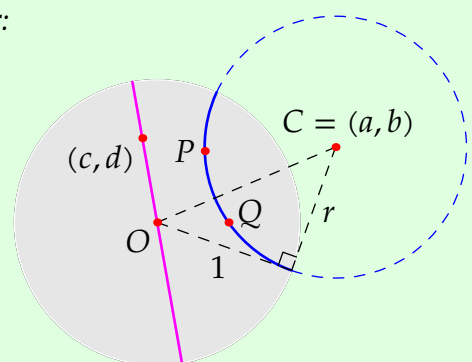
By the incidence axioms, there exists a unique hyperbolic line joining any two points in the Poincaré disk. Such may straightforwardly be described using equations in analytic geometry.

Lemma 4.8. A hyperbolic line in the Poincaré disk model is either:

1. A **diameter** with Euclidean equation $dx = cy$ (passes through a point $(c, d) \neq (0, 0)$).
2. An **arc** of a Euclidean circle with equation

$$x^2 + y^2 - 2ax - 2by + 1 = 0 \quad \text{where} \quad a^2 + b^2 > 1$$

Euclidean center $C = (a, b)$ and radius $r = \sqrt{a^2 + b^2 - 1}$.



²⁸Bolyai is the source of the term ‘absolute geometry.’

²⁹The key results of hyperbolic geometry can be discussed synthetically without reference to a model, though such an approach would be ahistorical, and masochistic for a first exposure!

Example 4.9. We compute the equation of the hyperbolic line through the points $P = (\frac{1}{3}, \frac{1}{2})$ and $Q = (\frac{1}{2}, 0)$ in the Poincaré disk: this is the picture shown in Lemma 4.8.

This requires only high-school algebra! Substitute the given points into the equation for a hyperbolic line ($x^2 + y^2 - 2ax - 2by + 1 = 0$) to obtain equations for the coefficients a, b :

$$\begin{cases} \frac{1}{9} + \frac{1}{4} - \frac{2}{3}a - b + 1 = 0 \\ \frac{1}{4} - a + 1 = 0 \end{cases} \implies (a, b) = \left(\frac{5}{4}, \frac{19}{36}\right)$$

The required hyperbolic line \overleftrightarrow{PQ} therefore has equation

$$x^2 + y^2 - \frac{5}{2}x - \frac{19}{18}y + 1 = 0 \quad \text{or} \quad \left(x - \frac{5}{4}\right)^2 + \left(y - \frac{19}{36}\right)^2 = \frac{545}{648}$$

This is an arc of a Euclidean circle centered at $C = (\frac{5}{4}, \frac{19}{36})$ with radius $r = \sqrt{\frac{545}{648}}$.

Distance and Angle in the Poincaré Disk The undefined terms *point*, *line*, *on* and *between* now make sense. To complete the model, we need to define *congruence* of segments and angles.

Definition 4.10. The *hyperbolic distance* between points P, Q in the Poincaré disk is³⁰

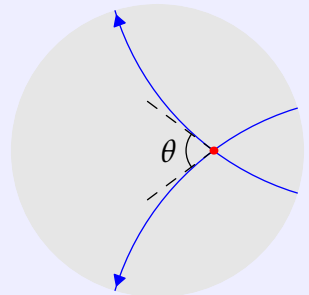
$$d(P, Q) := \cosh^{-1} \left(1 + \frac{2|PQ|^2}{(1 - |P|^2)(1 - |Q|^2)} \right)$$

where $|PQ|$, $|P| = |OP|$ and $|Q| = |OQ|$ are Euclidean distances. Hyperbolic segments are *congruent* if they have the same length

$$\overline{AB} \cong \overline{PQ} \iff d(A, B) = d(P, Q)$$

The *angle* between hyperbolic rays is that between their Euclidean tangent rays. Angles are *congruent* if they are so in a Euclidean sense.

The *hyperbolic circle* with center P and (hyperbolic) radius ρ is now easily defined as the set of points X for which $d(X, P) = \rho$.



It is worth observing both the symmetry of the distance function ($d(P, Q) = d(Q, P)$) and how it simplifies when measuring from the origin: if $Q = O$, then

$$d(O, P) = \cosh^{-1} \frac{1 + |P|^2}{1 - |P|^2} = \ln \frac{1 + |P|}{1 - |P|}$$

³⁰It should seem reasonable that hyperbolic functions to play some role in hyperbolic geometry! For reference:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh^2 x - \sinh^2 x = 1, \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$$

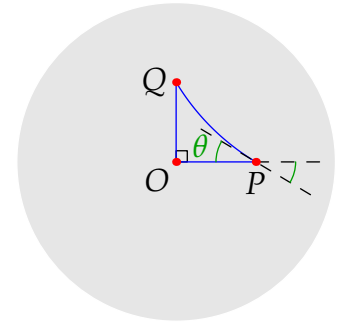
Example 4.11. We calculate the side-lengths and angles in the hyperbolic isosceles right-triangle with vertices $O = (0,0)$, $P = (\frac{1}{2},0)$ and $Q = (0,\frac{1}{2})$.

To compute the hyperbolic distances, we first need the three *Euclidean* versions:

$$|P| = \frac{1}{2} = |Q|, \quad |PQ|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$d(O,P) = d(O,Q) = \ln \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \ln 3 = \cosh^{-1} \frac{5}{3} \approx 1.10$$

$$d(P,Q) = \cosh^{-1} \left(1 + \frac{2 \cdot \frac{1}{2}}{(1 - \frac{1}{4})^2} \right) = \cosh^{-1} \frac{25}{9} \approx 1.68$$



To find the interior angle θ , implicitly differentiate the equation for the hyperbolic line \overleftrightarrow{PQ} :

$$x^2 + y^2 - \frac{5}{2}x - \frac{5}{2}y + 1 = 0 \implies \frac{dy}{dx} \Big|_P = \frac{4x - 5}{5 - 4y} \Big|_P = -\frac{3}{5} \implies \theta = \tan^{-1} \frac{3}{5} \approx 31^\circ$$

By symmetry, we have the same angle at Q . With a right-angle at O , we conclude that the angle-sum is $\Sigma_\Delta \approx 152^\circ$! To satisfy curiosity, we compare data for $\triangle OPQ$ and the *Euclidean* triangle with the same vertices

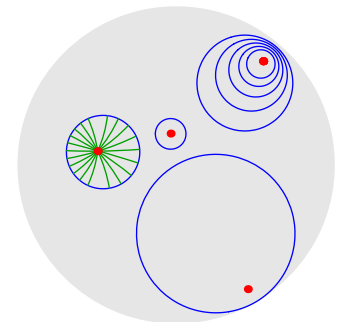
Property	Hyperbolic Triangle	Euclidean Triangle
Edge lengths	1.10 : 1.10 : 1.68	0.5 : 0.5 : 0.71
Relative edge ratios	1 : 1 : 1.53	1 : 1 : 1.41
Angles	31°, 31°, 90°	45°, 45°, 90°

The hyperbolic triangle has longer sides and a *relatively* longer hypotenuse. Moreover, its side lengths do *not* satisfy the Pythagorean relation³¹ $a^2 + b^2 = c^2$.

The Poincaré disk uses concepts from *Euclidean* (analytic) geometry to define objects and compute in *hyperbolic* geometry. It may seem laborious to keep stressing the distinction, but it is essential not to confuse the corresponding versions of distance, line, circle, etc. While the meaning of *angle* is unchanged, everything else is at least slightly, if not significantly, different.

For instance, the picture shows several **hyperbolic circles** and their **centers**; one has several of its **radii** drawn. Observe how the centers are closer (in a Euclidean sense) to the boundary circle (omega-points) than one might expect; this is since hyperbolic distance measure is greater the further one is from the origin.

In fact (Exercise 6) hyperbolic circles in the Poincaré disk model are also Euclidean circles! Their hyperbolic **radii** moreover meet the circles at right-angles, as we'd expect.



³¹Though the relationship $\cosh a \cosh b = \cosh c$ should be obvious. We'll return to this observation later...

A Little Bookkeeping The next result is an exercise; it says that the distance function behaves roughly as we expect, increasing smoothly as one moves along a hyperbolic line.

Lemma 4.12. Fix P and a hyperbolic line ℓ through P . Then the distance function $Q \mapsto d(P, Q)$ maps the points of ℓ on one side of P differentiably and bijectively onto the interval $(0, \infty)$.

This should also convince us that hyperbolic circles look roughly as we expect. Of greater importance is the following.

Theorem 4.13. The Poincaré disk is a model of hyperbolic geometry.

Sketch Proof. A rigorous proof would require us to check the hyperbolic postulate and all Hilbert’s axioms except Playfair. Instead we verify Euclid’s postulates 1–4 and the hyperbolic postulate 5.

1. Lemma 4.8 says we can join any given points in the Poincaré disk by a unique segment.
2. A hyperbolic segment joins two points *inside* the (open) Poincaré disk. The distance formula increases (Lemma 4.12) unboundedly as P moves towards the boundary circle, so we can always make a hyperbolic segment longer.
3. Hyperbolic circles are as in Definition 4.10.
4. All right-angles are equal since the notion of angle is unchanged from Euclidean geometry.
5. The first picture on page 69 shows multiple parallels! ■

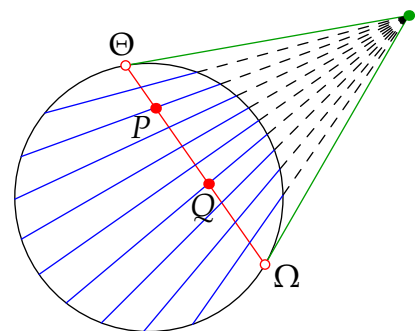
Other Models of Hyperbolic Space (non-examinable)

There are several other models of hyperbolic space. Here are three of the most common.

Klein Disk Model This is similar to the Poincaré disk. Instead of Euclidean arcs, hyperbolic lines are chords of the unit circle (‘Euclidean’ line-segments) without their end-points, and the distance function is different: if Ω, Θ are the omega-points of the chord \overleftrightarrow{PQ} , then

$$d_K(P, Q) = \frac{1}{2} \left| \ln \frac{|P\Theta||Q\Omega|}{|P\Omega||Q\Theta|} \right|$$

Lines might look “straight”, but the cost is that the notion of *angle* is different.³² Given a **hyperbolic line** (as pictured), find the **tangents** at its omega-points; any **chord** whose extension passes through the **intersection** of these tangents is perpendicular to the **original line**. Measuring other angles is difficult!

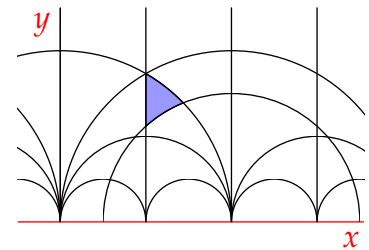


³²For differential geometers, Gauss’ famous *theorem egregium* says that this problem is unavoidable. The intrinsic (negative) curvature of hyperbolic geometry says that there is no model in which both lines and angles have the same meanings as in Euclidean geometry.

Poincaré Half-plane Model Widely used in complex analysis, the points comprise the upper half-plane ($y > 0$) in \mathbb{R}^2 , while hyperbolic lines are verticals or semicircles centered on the x -axis

$$x = \text{constant} \quad \text{or} \quad (x - a)^2 + y^2 = r^2$$

and angles are the same as in Euclidean space. The expression for hyperbolic distance remains horrific! The picture shows several hyperbolic lines and a **hyperbolic triangle**.

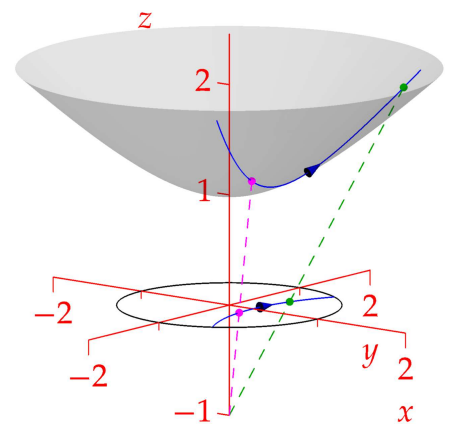


Hyperboloid Model Points comprise the upper sheet ($z \geq 1$) of the hyperboloid $x^2 + y^2 = z^2 - 1$. A **hyperbolic line** is the intersection of the hyperboloid with a plane through the origin. Isometries (congruence) can be described using matrix-multiplication and hyperbolic distance is relatively easy: given $P = (x, y, z)$ and $Q = (a, b, c)$,

$$d(P, Q) = \cosh^{-1}(cz - ax - by)$$

An obvious drawback is the need to work in three-dimensions. Moreover, angles are awkward.

The relationship to the Poincaré disk is via projection. Place the disk in the xy -plane centered at the origin and draw a **line** through the disk and the point $(0, 0, -1)$. The intersection of this line with the hyperboloid gives the correspondence.



Exercises 4.2. Key concepts: Hyperbolic Postulate, Poincaré disk, Hyperbolic lines and distance

Answer all questions within the Poincaré disk model.

1. (a) Find the equation of the hyperbolic line through $P = (\frac{1}{4}, 0)$ and $Q = (0, \frac{1}{2})$.
 (b) Find the side lengths of the hyperbolic triangle $\triangle OPQ$ where $O = (0, 0)$ is the origin.
 (c) $\triangle OPQ$ is right-angled at O . If o, p, q represent the hyperbolic lengths of the sides opposite O, P, Q respectively, check that $p^2 + q^2 \neq o^2$ is *false*. Now compute $\cosh p \cosh q$: what do you observe?
2. Find the omega points for the hyperbolic line with equation $x^2 + y^2 - 4x + 10y + 1 = 0$.
3. Let $P = (\frac{1}{2}, \sqrt{\frac{5}{12}})$ and $Q = (\frac{1}{2}, -\sqrt{\frac{5}{12}})$. Together with the origin O these form a triangle.
 - (a) Compute the hyperbolic distances $d(O, P)$, $d(O, Q)$ and $d(P, Q)$.
 - (b) Compute the interior angle at O ($\sphericalangle POQ$).
 - (c) Show that the hyperbolic line $\ell = \overline{PQ}$ has equation $x^2 - \frac{10}{3}x + y^2 + 1 = 0$.
 - (d) Calculate $\frac{dy}{dx}$ to show that a tangent vector to ℓ at P is $\sqrt{15}\mathbf{i} + 7\mathbf{j}$. Hence compute the interior angle at P ($\sphericalangle OPQ$). What is the angle-sum in this triangle?

4. We generalize Example 4.11. Suppose $0 < c < 1$ and consider the isosceles right-triangle with vertices $O = (0, 0)$, $P = (c, 0)$ and $Q = (0, c)$ in the Poincaré disk.
- Compute the hyperbolic side lengths of $\triangle OPQ$ as functions of c .
 - Find the equation of the hyperbolic line joining $P = (c, 0)$ and $Q = (0, c)$.
 - Use implicit differentiation to prove that the interior angles at P and Q measure $\tan^{-1} \frac{1-c^2}{1+c^2}$. What happens as $c \rightarrow 0^+$ and as $c \rightarrow 1^-$?

5. Let $0 < r < 1$. Consider the hyperbolic triangle with vertices

$$P = (r, 0), \quad Q = \left(-\frac{r}{2}, \frac{\sqrt{3}r}{2}\right), \quad R = \left(-\frac{r}{2}, -\frac{\sqrt{3}r}{2}\right)$$

- Find the side-lengths and thus confirm that $\triangle PQR$ is equilateral.
 - Compute the interior angles of $\triangle PQR$ as functions of r .
 - What happens to the triangle (side-lengths, angles) as $r \rightarrow 0^+$? As $r \rightarrow 1^-$?
6. (a) Prove that the hyperbolic circle with hyperbolic radius $\rho = \ln 3$ and center $C = (\frac{1}{2}, 0)$ has *Euclidean* equation

$$\left(x - \frac{2}{5}\right)^2 + y^2 = \frac{4}{25}$$

- Prove that every hyperbolic circle in the Poincaré disk model is also a Euclidean circle. Conversely, what restrictions on the constants p, q, r guarantee that a Euclidean circle with equation $(x - p)^2 + (y - a)^2 = r^2$ describes a hyperbolic circle?
7. We sketch a proof of Lemma 4.12.

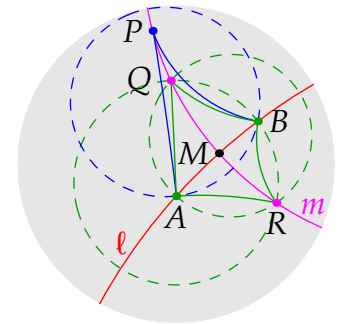
- Prove that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ is strictly increasing on the interval $(1, \infty)$.
- By part (a), it is enough to show that $\frac{|PQ|^2}{1-|Q|^2}$ increases as Q moves away from P along a hyperbolic line. Appealing to symmetry, let $P = (0, c)$ lie on the hyperbolic line with equation $x^2 + y^2 - 2by + 1 = 0$. If $Q = (x, y)$ also lies on the same line, prove that

$$\frac{|PQ|^2}{1-|Q|^2} = \frac{(b-c)y + bc - 1}{1-by}$$

and show that this is an increasing function of y when $c < y < \frac{1}{b}$.

4.3 Parallels, Perpendiculars & Angle-Sums

From now on, all examples will be illustrated within the Poincaré disk, though the main results hold in any model. Recall (page 65) that we may use anything from absolute geometry; as a sanity check, think through how the picture illustrates the following familiar result.

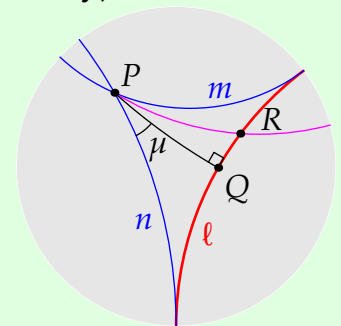


Lemma 4.14. Through a point P not on a line ℓ there exists a unique perpendicular to ℓ .

We now consider a major departure from Euclidean geometry.

Theorem 4.15 (Fundamental Theorem of Parallels in Hyperbolic Geometry).

Given $P \notin \ell$, drop the perpendicular \overline{PQ} to ℓ . Then there exist precisely two **limiting**, or **asymptotic**, **parallels** m, n to ℓ through P with the following properties:



1. A ray based at P intersects ℓ if and only if it lies between m and n in the same fashion as \overrightarrow{PQ} .
2. The limiting parallels make congruent acute angles μ with \overrightarrow{PQ} , the **angle of parallelism** at P relative to ℓ .

Every other (non-asymptotic) parallel is called an *ultraparallel*. More generally, parallel lines ℓ, m are *asymptotic* if they ‘meet’ at an omega-point.

The proof depends crucially on ideas from analysis, particularly continuity & suprema. As you read through, consider how everything *except* the last line is valid in Euclidean geometry!

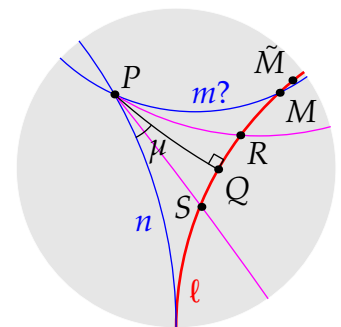
Proof. Points $R \in \ell$ are in continuous bijective correspondence with the real numbers (Lemma 4.12). We therefore have a *continuous, increasing* function

$$f : \mathbb{R} \rightarrow (-90^\circ, 90^\circ) \quad \text{where} \quad f(r) = \sphericalangle QPR \quad (\pm 90^\circ \notin \text{range } f \text{ by Saccheri–Legendre})$$

Since $\text{dom } f = \mathbb{R}$ is an interval, the intermediate value theorem says that $\text{range } f$ is a *subinterval* $I \subseteq (-90^\circ, 90^\circ)$.

Given $R \in \ell$, transfer \overline{QR} to the other side of Q to obtain $S \in \ell$. By SAS, $\sphericalangle QPS = -\sphericalangle QPR$ whence I is *symmetric*: $\theta \in I \iff -\theta \in I$.

Let $\mu := \sup I \in (0^\circ, 90^\circ]$ be the least upper bound; by symmetry, $\inf I = -\mu$. Let m and n be the lines making angles $\pm\mu$ respectively. Certainly any ray between m, n intersects ℓ . To finish part 1, it remains only to see that m, n are parallel to ℓ (that is, $I = (-\mu, \mu)$).



For contradiction, suppose m intersects ℓ at M . Transfer \overline{QM} to the other side of M to obtain \tilde{M} . Since f is increasing, $\sphericalangle QPM\tilde{M} > \mu = \sphericalangle QPM$ contradicts $\mu = \sup I$.

Finally $m = n \iff \mu = 90^\circ$. In such a case there would exist only one parallel to ℓ through P , contradicting the hyperbolic postulate. ■

The picture suggests a bijective relationship between μ and the perpendicular distance. Here it is; a simplified argument is in Exercise 4.3.6, and the full result to the next section.

Corollary 4.16. *The perpendicular distance $\delta = d(P, Q)$ and the angle of parallelism are related via*

$$\cosh \delta = \csc \mu \quad \text{or equivalently} \quad \tan \frac{\mu}{2} = e^{-\delta}$$

Examples 4.17. We compute the limiting parallels and angles of parallelism in two cases.

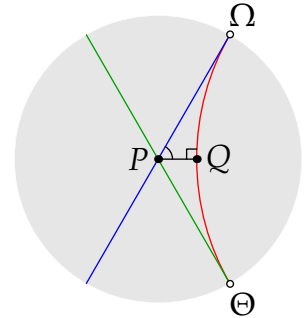
- Let ℓ be the hyperbolic line with equation $x^2 + y^2 - 4x + 1 = 0$. Intersect with $x^2 + y^2 = 1$ to find the omega-points $\Omega = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\Theta = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

By symmetry, the perpendicular \overline{PQ} from the origin P has equation $y = 0$, resulting in $Q = (2 - \sqrt{3}, 0)$.

The limiting parallels through P have equations $y = \pm\sqrt{3}x$, from which the angle of parallelism is $\mu = \tan^{-1} \sqrt{3} = 60^\circ$.

In accordance with Corollary 4.16, we easily verify that

$$\delta = d(P, Q) = \ln \frac{1 + (2 - \sqrt{3})}{1 - (2 - \sqrt{3})} = \ln \sqrt{3} \Leftrightarrow e^{-\delta} = \frac{1}{\sqrt{3}} = \tan \frac{\mu}{2}$$



- Let $P = \left(-\frac{3}{10}, \frac{4}{10}\right)$ relative to the **hyperbolic line** with equation $x^2 + y^2 + 2x + 4y + 1 = 0$. First find the omega-points by intersecting with $x^2 + y^2 = 1$:

$$\Omega = (-1, 0), \quad \Theta = \left(\frac{3}{5}, -\frac{4}{5}\right)$$

Plainly $\overline{P\Theta}$ is the diameter $y = -\frac{4}{3}x$ with slope $-\frac{4}{3}$.

For $\overline{P\Omega}$, substitute these points into the usual expression for a hyperbolic line (Lemma 4.8); implicitly differentiate for the **slope**:

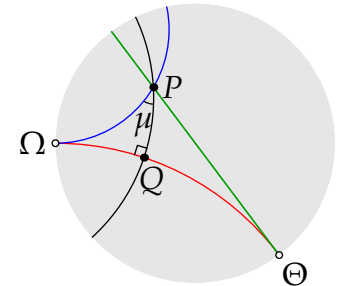
$$x^2 + y^2 + 2x - \frac{13}{8}y + 1 = 0 \Rightarrow \left. \frac{dy}{dx} \right|_P = \frac{16(1+x)}{13-16y} \Big|_P = \frac{16 \cdot \frac{7}{10}}{13 - \frac{64}{10}} = \frac{56}{33}$$

The angle of parallelism is *half* that between the tangent vectors $\begin{pmatrix} -33 \\ -56 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$:

$$\mu = \frac{1}{2} \cos^{-1} \frac{\begin{pmatrix} -33 \\ -56 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}}{\left| \begin{pmatrix} -33 \\ -56 \end{pmatrix} \right| \left| \begin{pmatrix} 3 \\ -4 \end{pmatrix} \right|} = \frac{1}{2} \cos^{-1} \frac{5}{13} \approx 33.7^\circ$$

Corollary 4.16 can now be used to find the perpendicular distance $d(P, Q) = \ln \frac{3+\sqrt{13}}{2}$.

If you want a major challenge, try to verify that $Q = \left(\frac{93(-29+2\sqrt{117})}{1865}, \frac{26(-29+2\sqrt{117})}{1865}\right)$.

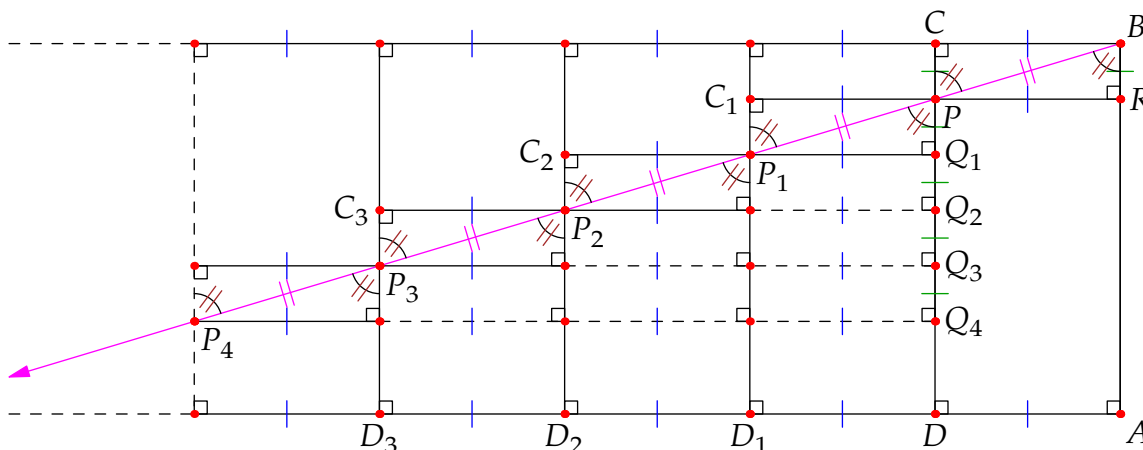


Angles in Triangles, Rectangles and the AAA Congruence

We at last come to the triple punchline. Everything follows from one big picture.

Theorem 4.18. Suppose $\square ABCD$ is a rectangle in absolute geometry. Then the angle of parallelism of B with respect to $\ell := \overrightarrow{AD}$ is 90° . Otherwise said, \overrightarrow{BC} is the unique parallel to \overrightarrow{AD} through B .

Proof. Reflect $\square ABCD$ across \overline{CD} and repeat to obtain an infinite family of congruent rectangles (Exercise 4.1.4). Let $P \in \overline{CD}$ and drop perpendiculars to $R \in \overline{AB}$ and C_1 as shown.



We claim $\square PRBC$ is a rectangle: if not, then one of $\square ARPD$ or $\square PRBC$ would have angle-sum exceeding 360° and contradicting Saccheri–Legendre. Similarly $\square DPC_1D_1$ is a rectangle.

By Exercise 4.1.3, \overline{BP} splits $\square PRBC$ into a pair of congruent triangles. In particular, \overline{BP} crosses \overline{CD} at the **same angle** it leaves B .

Iterate to obtain the picture: the small rectangles are congruent (essentially Exercise 4.1.4 again), whence P, Q_1, Q_2, \dots is an *equidistant* sequence along \overline{CD} .

Since \overline{CD} is *finite*, the Archimedean property (Exercise 2.4.8) says that this sequence eventually passes D : some Q_n lies on the opposite side of ℓ to B , whence $P_n \in \overline{BP}$ does also. We conclude that \overline{BP} intersects ℓ .

Since $P \in \overline{CD}$ was generic, the angle of parallelism of B with respect to ℓ is 90° . ■

Corollary 4.19. Within hyperbolic geometry:

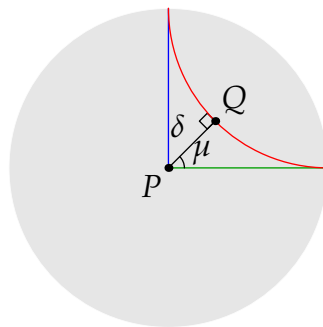
1. There are **no rectangles** (quadrilaterals with four right-angles). In particular, the summit angles of a Saccheri quadrilateral are **acute** (compare Theorem 4.5).
2. The angles in a triangle sum to **strictly less than 180°** .
3. (AAA congruence) If the angles of $\triangle ABC$ and $\triangle DEF$ are congruent in pairs, then the triangles are **congruent** ($\triangle ABC \cong \triangle DEF$).

Part 1 is simply the Theorem: a unique parallel contradicts the hyperbolic postulate. We leave the rest to Exercises 7 and 8. Notice that AAA is a *congruence* theorem in hyperbolic geometry,

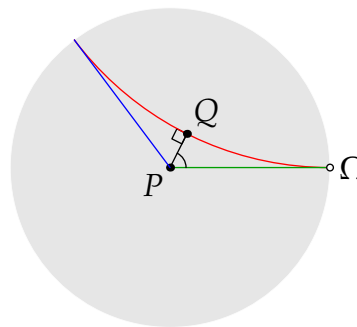
not a *similarity* theorem (compare Theorem 2.43). Finally, revisit the observations on page 65; Euclid's arguments making use of the parallel postulate genuinely *require* it!

Exercises 4.3. *Key concepts: Limiting parallels, Angle of parallelism, $\cosh \delta = \csc \mu$, No rectangles, $\Sigma_{\Delta} < 180^\circ$, AAA congruence*

1. Use Corollary 4.19 to prove the following within hyperbolic geometry.
 - (a) Two hyperbolic lines cannot have more than one common perpendicular.
 - (b) Saccheri quadrilaterals with congruent summits and summit angles are congruent.
2. Suppose P lies a perpendicular distance $\delta = d(P, Q) = \ln \sqrt{3} = \frac{1}{2} \ln 3$ from a hyperbolic line ℓ , and that the ray \overrightarrow{PR} makes angle 45° with the perpendicular \overrightarrow{PQ} . Determine whether \overrightarrow{PQ} intersects ℓ , is a limiting parallel, or an ultraparallel.
3. Suppose ℓ intersects m at a right-angle and that m, n are parallel.
 - (a) In *Euclidean geometry*, prove that ℓ intersects n at a right-angle.
 - (b) What are the possible arrangements in *hyperbolic geometry*? Draw some pictures.
4. Let ℓ be the hyperbolic line with equation $(x - 1)^2 + (y - 1)^2 = 1$, let P be the origin and drop the perpendicular \overrightarrow{PQ} to $Q \in \ell$.
 - (a) Explain why the angle of parallelism of P with respect to ℓ is 45° and why Q has co-ordinates $(1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}})$.
 - (b) Use the hyperbolic distance formula (Definition 4.10) to compute the perpendicular distance from ℓ to the origin. Hence verify the relation $\cosh \delta = \csc \mu$.



Question 4



Question 5

5. Similarly to Question 4, verify the relation $\cosh \delta = \csc \mu$ for the origin P relative to the hyperbolic line ℓ with equation $x^2 + y^2 - 2x - 4y + 1 = 0$.
 (*Hints: Where is the 'center' of ℓ viewed as a Euclidean circle? As in Question 4, $\Omega = (1, 0)$ is an omega-point for ℓ .)*)

6. We generalize Example 4.17.1. Suppose $P = (0, 0)$ is the origin and let $Q = (r, 0)$ where $0 < r < 1$. Let ℓ be the hyperbolic line passing through Q at right-angles to \overline{PQ} .

(a) Find the equation of ℓ and prove that its limiting parallels through P have equations

$$\pm 2ry = (1 - r^2)x$$

(b) Let μ be the angle of parallelism of P relative to ℓ and $\delta = d(P, Q)$ the hyperbolic distance. Prove the special case of Corollary 4.16, that $\cosh \delta = \csc \mu$.

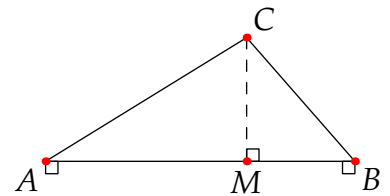
(Hint: $\csc^2 \mu = 1 + \cot^2 \mu = 1 + \frac{1}{\tan^2 \mu} = \dots$)

(c) By differentiating $\cosh \delta = \csc \mu$, verify the claim that δ and μ are bijectively related.

7. We work in absolute geometry. Suppose $\triangle ABC$ has longest side \overline{AB} (the other sides are no larger) and drop the perpendicular from C to $M \in \overline{AB}$.

(a) Prove that M is interior to \overline{AB} by showing that the other possibilities are contradictions.

(b) Suppose a triangle exists with angle-sum 180° . Show that there exists such a *right-triangle*, and therefore a rectangle. Hence conclude part 2 of Corollary 4.19.



(c) Explain why part (a) is needed to prove (b): what might happen if \overline{AB} isn't the longest side?

8. We prove the AAA congruence theorem in hyperbolic geometry (Corollary 4.19, part 3).

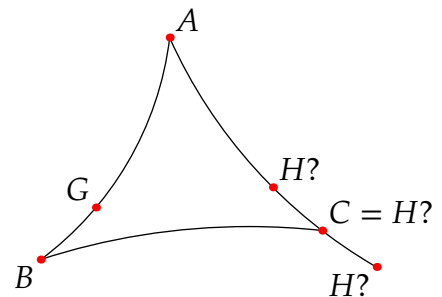
Suppose, for contradiction, that *non-congruent* triangles $\triangle ABC$ and $\triangle DEF$ have angles congruent in pairs ($\angle A \cong \angle D$, etc.). Without loss of generality, assume $\overline{DE} < \overline{AB}$. By segment transfer, there exist unique points:

- G interior to \overline{AB} such that $\overline{DE} \cong \overline{AG}$.
- $H \in \overline{AC}$ such that $\overline{DF} \cong \overline{AH}$.

(a) Explain why $\triangle DEF \cong \triangle AGH$.

(b) There are three possible locations for H .

- i. H is interior to \overline{AC} .
- ii. $H = C$.
- iii. C lies between A and H .



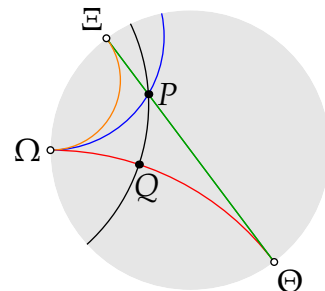
By connecting \overline{GH} , in each case explain why we have a contradiction.

4.4 Omega-triangles

Recall that limiting parallels (Theorem 4.15) ‘meet’ at an omega-point.

Definition 4.20. An *omega-triangle* or *ideal-triangle* is a ‘triangle’ where at least two sides are limiting parallels. Alternatively (in the Poincaré disk model), one or more of the vertices is an omega-point.

The three types of omega-triangle depend on how many omega-points they have. In the picture, $\triangle PQ\Omega$ has one omega-point, $\triangle P\Omega\Theta$ has two, and $\triangle \Omega\Theta\Xi$ three!



Amazingly, many of the standard results of absolute geometry also apply to omega-triangles! The first can be thought of as the AAA congruence theorem where one ‘angle’ is zero.

Theorem 4.21 (Angle-Angle Congruence for Omega-triangles). Suppose $\triangle AB\Omega$ and $\triangle PQ\Theta$ are omega-triangles, each with a single omega-point. If the (finite) angles are congruent in pairs

$$\angle AB\Omega \cong \angle PQ\Theta \quad \angle BA\Omega \cong \angle QP\Theta$$

then the finite sides of the omega-triangles are also congruent: $\overline{AB} \cong \overline{PQ}$.

Remember that omega-points are not really part of hyperbolic geometry—their appearance in our description is a quirk of the Poincaré disk model. It therefore doesn’t make much sense to speak of congruent infinite sides or of congruent angles at omega-points.

Proof. WLOG and for contradiction, assume $\overline{AB} > \overline{PQ}$. Transfer $\angle QP\Theta$ to A to obtain $C \in \overline{AB}$ such that $\overline{AC} \cong \overline{PQ}$. Transferring $\angle PQ\Theta$ creates a ray r based at C on the same side as Ω . Exercise 3 verifies $r = \overline{C\Omega}$. Our hypothesis is that the pictured angles at B and C are congruent.

Let M be the midpoint of \overline{BC} and drop the perpendicular to $R \in \overline{B\Omega}$.

Let $S \in \overline{C\Omega}$ lie on the opposite side of \overline{BC} to R such that $\overline{CS} \cong \overline{BR}$.

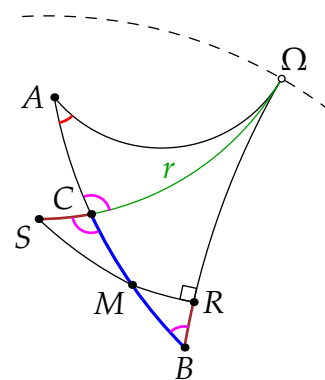
By **Side-Angle-Side** we have $\triangle MBR \cong \triangle MCS$. In particular:

- $\triangle MCS$ is right-angled(!) at S .
- Congruent vertical angles at M force M to lie on the segment \overline{RS} .

The angle of parallelism of S relative to $\overline{B\Omega}$ is therefore $\angle RS\Omega = 90^\circ$, contradicting the Fundamental Theorem (4.15).

There are two other possible orientations:

- R could lie on the opposite side of B from Ω . In this case SAS is applied to the same triangles but with respect to congruent **supplementary angles**.
- In the special case $R = B$, the same contradiction appears: the angle of parallelism of C with respect to $\overline{B\Omega}$ is 90° . ■



Theorem 4.22 (Exterior Angle Theorem for Omega-Triangles). Suppose $\triangle BC\Omega$ has a single omega-point and that C lies between A and B . Then $\angle AC\Omega > \angle CB\Omega$.

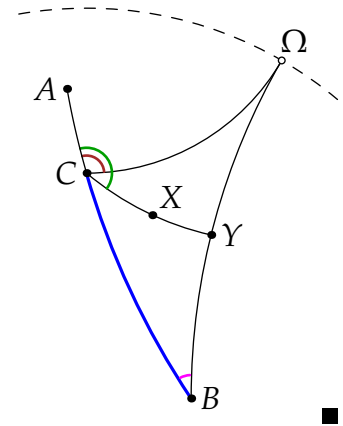
Proof. We show that the two other cases are impossible.

($\angle AC\Omega \cong \angle CB\Omega$) This is precisely the contradictory arrangement described in the previous proof!

($\angle AC\Omega < \angle CB\Omega$) Transfer the latter to C to produce \overrightarrow{CX} interior to $\angle BC\Omega$ with $\angle ACX \cong \angle CB\Omega$.

Since $\overrightarrow{C\Omega}$ is a limiting parallel to $\overrightarrow{C\Omega}$, the Fundamental Theorem says that \overrightarrow{CX} intersects $\overrightarrow{B\Omega}$ at some point Y .

But now $\triangle CBY$ contradicts the standard exterior angle theorem ($\angle ACY \cong \angle CBY$).



Corollary 4.23 (Side-Angle Congruence for Omega-triangles). Suppose $\triangle BC\Omega$ and $\triangle PQ\Theta$ both have a single omega-point. If $\angle CB\Omega \cong \angle QP\Theta$ and $\overline{BC} \cong \overline{PQ}$ then $\angle BC\Omega \cong \angle PQ\Theta$.

The final congruence theorem is an exercise based on the previous picture. In summary: A triangle with a single omega-point has three pieces of data (two finite angles and one edge). The AA and SA congruence theorems say that two of these determine the third.

Other observations

Pasch's Axiom & the Crossbar Theorem Versions of these are *theorems* for omega-triangles.

- If a line crosses a side of an omega-triangle and does not pass through any vertex (including Ω), then it must pass through exactly one of the other sides.
- If a line passes through an interior point and exactly one vertex (including Ω) of an omega-triangle, then it passes through the opposite side.

Perpendicular Distance and the Angle of Parallelism Applied to right-angled omega-triangles, the AA and SA theorems prove that the angle of parallelism is a bijective function of the perpendicular distance. Moreover, transferring the right-angle to the positive x -axis and the other vertex to the origin produces the arrangement in Exercise 4.3.6. This completes the proof of Corollary 4.16.

Exercises 4.4. Key concepts: Omega-triangle, AA & SA congruences

1. Let $\triangle PQ\Omega$ be an omega-triangle. Prove that $\angle PQ\Omega + \angle QP\Omega < 180^\circ$.
2. Let ℓ and m be limiting parallels. Explain why they cannot have a common perpendicular.
3. In the proof of the AA congruence, explain why r cannot intersect either $\overline{A\Omega}$ or $\overline{B\Omega}$.
4. Prove the Side-Angle congruence theorem for omega-triangles with one omega-point.
5. What would an 'omega-triangle' look like in Euclidean geometry? Comment on the three results in this section: are they still true?