5 Fractal Geometry

5.1 Natural Geometry, Self-similarity and Fractal Dimension

The objects of classical geometry (lines, curves, spheres, etc.) seem flatter and less interesting as one zooms in: at small scales, every differentiable curve looks like a line segment! By contrast, real-world objects tend to exhibit greater detail at smaller scales. A seemingly spherical orange is dimpled on closer inspection: is its surface area that of a sphere, or is the area greater due to the dimples? What if we zoom in further? Under a microscope, the dimples are seen to have minute cracks and fissures. With modern technology, we can 'see' almost to the molecular level; what does *surface area* even mean at such a scale?

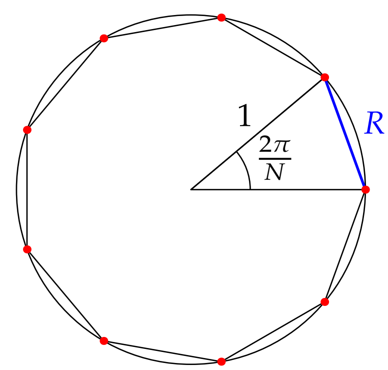
The Length of a Coastline

In 1967 Benoit Mandelbrot asked a related question in a now-famous paper, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension.* His essential point was that this question has no simple answer:[42] Should we measure by walking along the mean high tide line? Do we include the boundary of every small rock? Do we skirt every grain of sand? Every molecule? As the scale of consideration shrinks, the measured length becomes larger.

Example 5.1. Mandelbrot's approach is little more than the standard method for measuring the arc-length of a curve in calculus. Here it is applied to a smooth circular 'coastline.'

Approximate the path using N rulers of length R: clearly

$$R = 2\sin\frac{\pi}{N}$$



The sum of the ruler-lengths approximates the circumference:

$$NR = 2N\sin\frac{\pi}{N} \approx 2N\cdot\frac{\pi}{N} = 2\pi$$

where we used the small angle approximation for sine[43] as $N \to \infty$.
Otherwise said, as the length R of the measuring ruler shrinks, the number required to measure the path grows according to

$$N \approx 2\pi R^{-1}$$

There is nothing special about the circle. Measuring any smooth "calculus" curve with N small rulers of length R, we see that the arc-length is approximately $L \approx NR$ (Exercise 3).

But what happens if we apply this approach to a non-smooth curve like a coastline…

[42] The official answer from the Ordnance Survey (the UK government's mapping office) is, 'It depends.' The all-knowing CIA states 7723 miles, though offers no evidence as to why.

[43] This is the familiar fact that $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Definition 5.2 (Fractal Dimension of a Curve (empirical)). Given a ruler of length R, let N be the number of rulers needed to trace round a curve when laid end-to-end.

Plot $\log N$ against $\log(1/R)$ for several sizes of ruler. The data suggests a straight line:

$$\log N \approx \log k + D \log(1/R) = \log(kR^{-D}) \implies N \approx kR^{-D}$$

The number D is Mandelbrot's *fractal dimension* of the coastline.

Examples 5.3. While this notion of fractal dimension is purely empirical, it does seem to capture something about the 'roughness' of a curve.

1. Since a smooth curve with arc-length L satisfies $N \approx LR^{-1}$, any such reassuringly has fractal dimension $D = 1$.

2. Thanks to its relatively rugged north and west coasts, the island of Great Britain has fractal dimension significantly higher than that of a smooth curve $D \approx 1.25$.

British Mainland $D \approx 1.25$

Norwegian West Coast $D \approx 1.52$

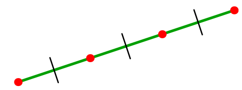
3. With its myriad fjords, Norway has an even rougher coastline,[44] and thus a higher fractal dimension $D \approx 1.52$.

For more detail regarding these coastline examples, see the Fractal Foundation's website. Mandelbrot coined the word *fractal*, though he didn't invent the concept from nothing. Rather he applied earlier ideas of Hausdorff, Minkowski and others, and observed how the natural world contains many examples of fractal structures.

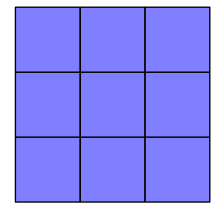
[44] Ignore the small islands, and the (smooth) eastern border with Sweden and Finland.

Self-Similar Figures

Our goal is to describe a related notion of fractional dimension. To help motivate this, recall some of the standard objects of pre-fractal geometry.



- A segment comprises N copies of itself each scaled by a factor $r = \frac{1}{N}$.
- A square comprises N copies of itself scaled by $r = \frac{1}{\sqrt{N}}$.
- A cube comprises N copies of itself scaled by $r = \frac{1}{\sqrt[3]{N}}$.

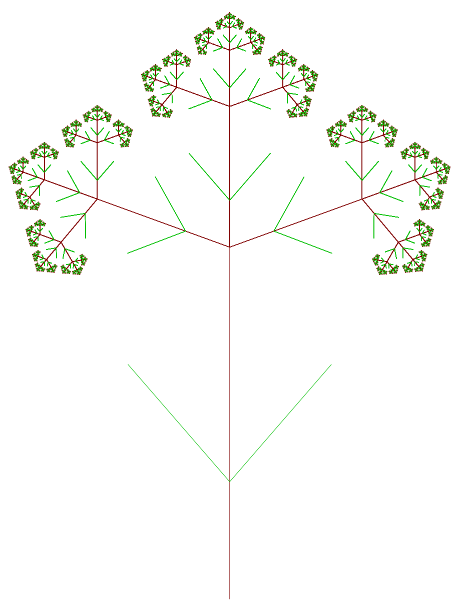


In each case $N = r^{-D}$, where D is the usual dimension $(1, 2$ or $3)$. Inspired by this, we make a loose definition.

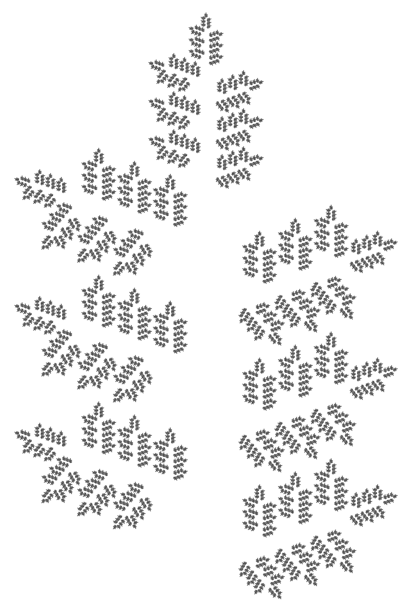
Definition 5.4. A geometric figure is *self-similar* if it may be subdivided[45] into N similar copies of itself, each scaled by a magnification factor $r < 1$. The *fractal dimension* of such a figure is

$$D := \log_{1/r} N = \frac{\log N}{\log(1/r)} = -\frac{\log N}{\log r}$$

Example 5.5. The pictures offer some evidence for non-integer fractal dimension, and for the idea that self-similarity is a natural phenomenon. If one ignores the stem, the 'tree' comprises $N = 3$ copies of itself each scaled by $r = 0.4$. The fern has $N = 7$ and $r = 0.3$.



Fractal Tree $D = -\frac{\log 3}{\log 0.4} \approx 1.20$



Fractal Fern $D = -\frac{\log 7}{\log 0.3} \approx 1.62$

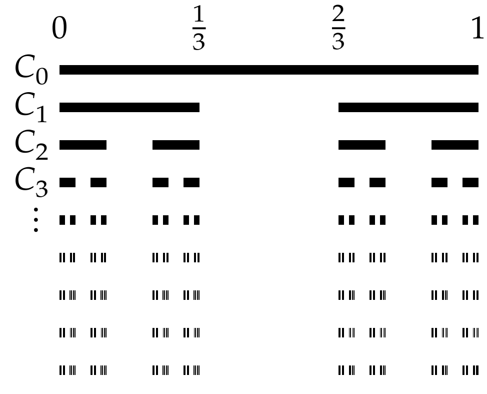
With dimensions between 1 and 2, both objects exhibit an intuitive idea of fractal dimension: both seem to occupy more space than mere *lines*, but neither has positive *area*. Moreover, the fern seems to occupy more space—has higher dimension—than the tree.

[45]In this and what follows, boundary points (e.g., endpoints of sub-segments) can end up in multiple subsets.

Example 5.6 (Cantor's Middle-Third Set). This famous example dates from the late 1800s.[46]

Starting with the unit interval $C_0 = [0,1]$, define a sequence of sets (C_n) where C_{n+1} is obtained by deleting the open 'middle-third' of each interval in C_n; for instance

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$



Cantor's set is essentially the limit of this sequence:

$$C := \bigcap_{n=0}^{\infty} C_n$$

Cantor's set has several strange properties, which we consider only non-rigorously.

Zero length Since we delete $\frac{1}{3}$ of the remaining set at each step, the sum of the lengths of the sub-intervals comprising C_n is $\left(\frac{2}{3}\right)^n$. It follows that

$$\forall n \in \mathbb{N}_0, \text{ length}(C) \le \left(\frac{2}{3}\right)^n \implies \text{length}(C) = 0$$

We conclude that C contains no subintervals!

Uncountability There exists a bijection between C and the original interval $[0,1]$. This is of limited interest to us, though you've likely encountered the notion elsewhere.

Self-similarity C_{n+1} comprises two copies of C_n, each shrunk by a factor of $\frac{1}{3}$. Abusing notation,

$$C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{1}{3}C_n + \frac{2}{3}\right) \xrightarrow[\text{'Take Limits'}]{} C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right)$$

Cantor's set is seen to comprise two shrunken copies of itself, and thus has fractal dimension $D = \frac{\log 2}{\log 3} \approx 0.63$.

Cantor's set has many generalizations:

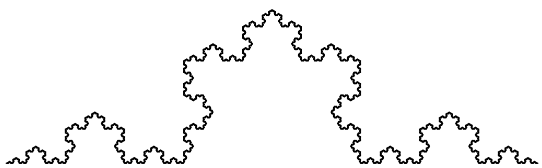
- Removing other fractions of each interval produces Cantor-type sets with other fractal dimensions. For instance, removing the 2nd and 4th fifths results in $D = \frac{\log 3}{\log 5} \approx 0.68$.
- Higher-dimensional analogues include the Sierpiński triangle ($D = \frac{\log 3}{\log 2} \approx 1.59$) and carpet (Example 5.12.3, $D = \frac{\log 8}{\log 3} \approx 1.89$), and the Menger sponge ($D = \frac{\log 20}{\log 3} \approx 2.73$).

[46]While the set is named for Cantor, it was first described during Henry Smith's 1874 investigations of integrability: this relates to the 'length' of a set, which was later formalized using *measure theory*.

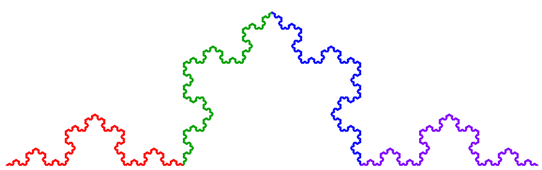
Example 5.7 (The Koch Curve and Snowflake). Another generalization of the Cantor set is produced as the limit of a sequence of curves.

- Let K_0 be a segment of length 1.
- Create K_1 by replacing the middle third of K_0 with the remaining sides of an upward-pointing equilateral triangle.
- Replace the middle third of each segment in K_1 as before to create K_2.
- Repeat *ad infinitum.*

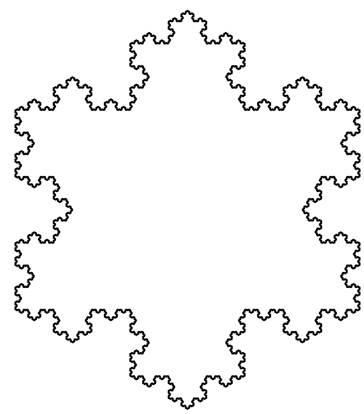
The resulting curve is drawn. The *Koch snowflake* is obtained by arranging three copies around an equilateral triangle.



Koch Curve



Self-similarity



Koch Snowflake

The relation to the Cantor set should be obvious in the construction. Indeed if we visualize K_0 as the interval $[0,1]$, then the intersection of K_0 with the Koch curve is the Cantor set itself!

The Koch curve is self-similar in that it comprises $N = 4$ copies of itself, each shrunk by a factor of $r = \frac{1}{3}$. Its fractal dimension is therefore

$$D = \frac{\log 4}{\log 3} \approx 1.26$$

We may also consider the curve's length. Let N_n be the number of segments in K_n, each having length R_n, then $\ell_n = N_n R_n$ is the length of the curve K_n. It follows that

$$N_n = 4^n, \quad R_n = \frac{1}{3^n} \implies \ell_n = \left(\frac{4}{3}\right)^n \to \infty$$

The Koch curve is *infinitely long*! Consider also how this construction fits with Mandelbrot's discussion of the fractal dimension of a coastline (Definition 5.2): measuring the Koch curve using N_n rulers of length R_n, we see that

$$\log N_n = n \log 4 = -\frac{\log 4}{\log 3} \log R_n \implies N_n = R_n^{-D}$$

Exercises 5.1. *Key concepts: 'Real' objects exhibit more detail when magnified, Self-similarity, Fractal Dimension* $D = -\frac{\log N}{\log r}$

1. By removing a constant middle fraction of each interval, construct a fractal analogous to the Cantor set but with dimension $\frac{1}{2}$.

   More generally, if one removes a constant middle fraction f from each interval, what is the resulting dimension?

2. Prove that the area inside the n^{th} iteration of the construction of the Koch snowflake is

$$A_n = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\left[1 - \left(\frac{4}{9}\right)^n\right]\right)$$

   What is the area inside the snowflake, and how does it relate to that of the original equilateral triangle on which it is based?

3. We prove that any regular (smooth) curve has fractal dimension 1 in the sense of Mandelbrot's definition (5.2).

   Suppose $\mathbf{r}(t)$, $t \in [0,1]$ parametrizes such a curve.

   (a) Use the arc-length formula $L = \int_0^1 |\mathbf{r}'(t)|\,\mathrm{d}t$ and linear approximations to $\mathbf{r}(t)$ to argue that,

$$L = \lim_{N\to\infty} \sum_{k=0}^{N-1} \left|\mathbf{r}\left(\frac{k+1}{N}\right) - \mathbf{r}\left(\frac{k}{N}\right)\right| \tag{$*$}$$

   (b) Suppose $\mathbf{r}(t)$ is parametrized such that each segment in $(*)$ has the same length R. Prove that for large N,

$$N \approx LR^{-1}$$

   whence the curve has fractal dimension 1.

5.2 Contraction Mappings & Iterated Function Systems

Thus far we have dealt informally with fractals in which the whole consists of multiple pieces scaled by the same factor. To add more variety, we'd like to consider multiple scale factors. To do this, and to be more rigorous, we need to borrow some ideas from topology and analysis.

Definition 5.8. A *contraction mapping* with scale factor $c \in [0,1)$ is a function $S : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\forall x, y \in \mathbb{R}^m, \; |S(x) - S(y)| \le c\,|x - y|$$

A contraction mapping moves inputs closer together. It should be clear that every contraction mapping is continuous ($\lim x_n = y \implies \lim S(x_n) = S(y)$).

Example 5.9. The function $S : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{1}{3}x + \frac{2}{3}$ is a contraction mapping with scale factor $c = \frac{1}{3}$: indeed

$$|S(x) - S(y)| = \frac{1}{3}\,|x - y|$$

To motivate the key theorem, consider using S to inductively define a sequence: given any x_0, define

$$x_{n+1} := S(x_n)$$

The first few terms are

$$x_1 = \frac{x_0}{3} + \frac{2}{3}, \qquad x_2 = \frac{x_0}{3^2} + \frac{2}{3^2} + \frac{2}{3}, \qquad x_3 = \frac{x_0}{3^3} + \frac{2}{3^3} + \frac{2}{3^2} + \frac{2}{3}$$

which suggests (geometric series)

$$x_n = \frac{x_0}{3^n} + 2\sum_{k=1}^{n} \frac{1}{3^k} = \frac{x_0}{3^n} + \frac{2(3^{-1} - 3^{-n-1})}{1 - 3^{-1}} = 1 + \frac{1}{3^n}(x_0 - 1)$$

This can easily be verified by induction if you prefer. The striking thing about this sequence is that it converges to the same limit $\lim x_n = 1$ *regardless of the initial term* x_0!

The example illustrates one of the most powerful and useful theorems in mathematics.

Theorem 5.10 (Banach Fixed Point Theorem). *Let $S : \mathbb{R} \to \mathbb{R}$ be a contraction mapping. Then:*

1. *S has a unique **fixed point**: some $L \in \mathbb{R}^m$ such that $S(L) = L$.*
2. *If $x_0 \in \mathbb{R}$ is **any** value, then the sequence defined iteratively by $x_{n+1} := S(x_n)$ converges to L.*

If you're comfortable with analysis, the proof is straightforward (Exercise 8).

In fact, as will be crucial momentarily, Banach's result holds whenever $S : \mathcal{H} \to \mathcal{H}$ is a contraction mapping on any *complete metric space*.[47] Our goal is to use Banach's result to generate certain types of fractal via repeated applications of contraction mappings to an initial shape. Our motivating example already illustrates this.

Example (5.6, Cantor Set mk. II). The functions $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ where

$$S_1(x) = \frac{x}{3} \qquad S_2(x) = \frac{x}{3} + \frac{2}{3}$$

are contraction mappings with scale factor $c = \frac{1}{3}$. More importantly, these functions *define* the Cantor set: at each stage of its construction, we defined

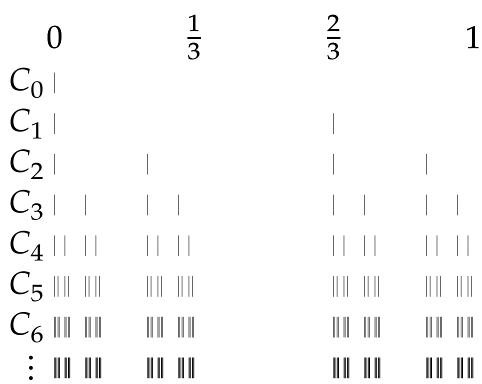
$$C_{n+1} := S_1(C_n) \cup S_2(C_n) \tag{$*$}$$

The self-similarity of the Cantor set can be expressed in the same manner: $\mathcal{C} = S_1(\mathcal{C}) \cup S_2(\mathcal{C})$.

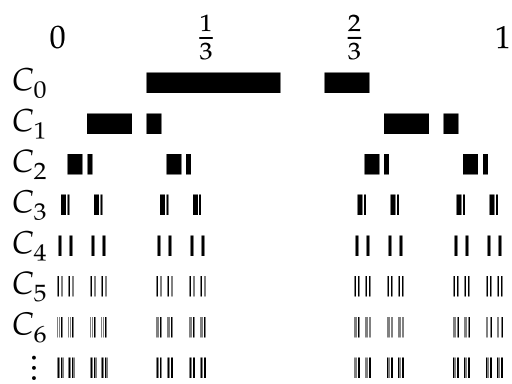
Amazingly, it barely seems to matter what initial set C_0 we choose. Originally, we took $C_0 = [0, 1]$ to be the unit interval, but we could instead start with the singleton set $C_0 = \{0\}$: in this case, iterating $(*)$ produces

$$C_1 = \{0, \tfrac{2}{3}\}, \quad C_2 = \{0, \tfrac{2}{9}, \tfrac{2}{3}, \tfrac{8}{9}\}, \quad C_3 = \{0, \tfrac{2}{27}, \tfrac{2}{9}, \tfrac{8}{27}, \tfrac{2}{3}, \tfrac{20}{27}, \tfrac{8}{9}, \tfrac{26}{27}\}, \quad \ldots$$

The first few iterations are drawn in the first picture below; it appears as if, in the limit, C_n is 'approaching' the Cantor set.



Initial set $C_0 = \{0\}$



Initial set $C_0 = [\frac{2}{10}, \frac{5}{10}] \cup [\frac{6}{10}, \frac{7}{10}]$

The second picture starts with a very different initial set $C_0 = [0.2, 0.5] \cup [0.6, 0.7]$; iterating this also appears to produce the Cantor set!

[47]Very loosely, a metric space is a set on which a sensible notion of *distance* can be defined: on Euclidean $\mathbb{R}^2$, for instance, the distance between two points is $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. If you've studied analysis you'll be familiar with *completeness*: every Cauchy sequence converges (in $\mathcal{H}$).

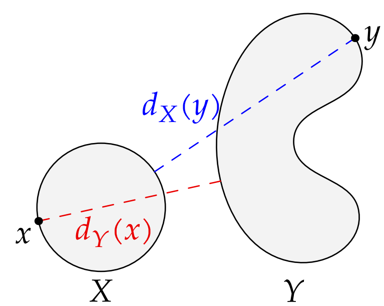
Iterated Function Systems

It certainly appears as if the Cantor set is generated by the contraction maps S_1, S_2 independently of the initial data C_0. Our main result shows in what sense this is true. Since this requires some heavy lifting from topology and analysis, we provide only a synopsis.

- A subset $K \subset \mathbb{R}^m$ is *compact* if it is *closed* (contains its boundary points) and *bounded* (K lies within some ball centered at the origin). For instance:
  - The closed interval $K = [0, 1]$ is a compact subset of $\mathbb{R}$.
  - The open disk $D = \{(x, y) : x^2 + y^2 < 1\}$ is a non-compact subset of $\mathbb{R}^2$, since it does not contain its boundary points (the unit circle).
- The set of non-empty compact subsets of $\mathbb{R}^m$ is a *metric space* $\mathcal{H}$. This means that the *distance* $d(X, Y)$ between $X, Y \in \mathcal{H}$ may sensibly be defined, though it is a little tricky...

  The distance function is the *Hausdorff metric*. Given $Y \in \mathcal{H}$ and $x \in \mathbb{R}^m$, define $d_Y(x) = \inf_{y \in Y} \|x - y\|$ to be the distance from x to the 'nearest' point of Y. Define $d_X(y)$ similarly. The distance between X and Y is then

  $$d(X, Y) := \max \left\{ \sup_{x \in X} d_Y(x), \sup_{y \in Y} d_X(y) \right\}$$

  Roughly speaking, find $x \in X$ which is as far away as possible from anything in Y, and find $y \in Y$ similarly; $d(X, Y)$ is the larger of these distances. Crucially, for compact sets, $d(X, Y) = 0 \iff X = Y$.
- Since $\mathcal{H}$ is a metric space, we can discuss convergent sequences (K_n) of compact sets

  $$\lim_{n \to \infty} K_n = K \iff \lim_{n \to \infty} d(K_n, K) = 0$$

  It also makes sense to speak of Cauchy sequences in $\mathcal{H}$. It may be proved that $\mathcal{H}$ is *complete*: every Cauchy sequence $(K_n) \subseteq \mathcal{H}$ converges to some $K \in \mathcal{H}$.
- The main result is a corollary of Banach's Theorem (5.10).

Theorem 5.11. *Let $S_1, \ldots, S_n$ be contraction mappings on $\mathbb{R}^m$ with scale factors $c_1, \ldots, c_n$. Define*

$$S : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad S(K) = \bigcup_{i=1}^{n} S_i(K)$$

1. *S is a contraction mapping on $\mathcal{H}$ with scale factor $c = \max(c_1, \ldots, c_n)$.*
2. *S has a unique fixed set $F \in \mathcal{H}$ given by $F = \lim_{k \to \infty} S^k(K_0)$ for **any** non-empty $K_0 \in \mathcal{H}$.*

Part 1 is not super difficult to prove if you are willing to grapple with the Hausdorff metric (try it if you're comfortable with analysis). Part 2 is Banach's theorem.

The upshot is this: repeatedly applying contraction mappings to *any* non-empty compact set K_0 produces a compact limit set which is *independent of* K_0! We call this limit F for *fractal*. Such fractals are sometimes called *attractors*: being limit-sets, they 'attract' data towards themselves.

Examples 5.12. We consider four iterated function systems.

1. (Example 5.6, mk.III) The contractions $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ (on $\mathbb{R}$) produce a contraction map $S : \mathcal{H} \to \mathcal{H}$:

$$S(K) := \{S_1(x), S_2(x) : x \in K\}$$

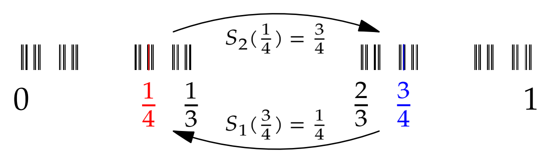
By Theorem 5.11, if $C_0 \subset \mathbb{R}$ is non-empty, closed and bounded, then $\mathcal{C} = \lim S^n(C_0)$. Certainly all three of our previous choices for C_0 are such sets, namely $[0,1]$, $\{0\}$ and $[0.2, 0.5] \cup [0.6, 0.7]$.

A fun application of the Theorem allows us to find all sorts of interesting points in the Cantor set. For instance, consider the compositions T and U, where

$$T(x) = S_1\big(S_2(x)\big) = \frac{x}{9} + \frac{2}{9} \quad \text{and} \quad U(y) = S_2\big(S_1(y)\big) = \frac{y}{9} + \frac{2}{3}$$

These are both contraction maps on $\mathbb{R}$ ($c = \frac{1}{9}$) whose unique fixed points are $t = \frac{1}{4}$ and $u = \frac{3}{4}$; moreover $S_1(u) = t$ and $S_2(t) = u$. Now consider the non-empty compact set $K = \{t, u\} \in \mathcal{H}$. Plainly

$$\begin{aligned} S(K) &= \{S_1(t), S_1(u), S_2(t), S_2(u)\} \\ &= \left\{\frac{1}{12}, \frac{1}{4}, \frac{3}{4}, \frac{11}{12}\right\} \supset K \end{aligned}$$



It follows (by induction and Theorem 5.11) that $K \subseteq \lim S^n(K) = \mathcal{C}$: otherwise said, both $t = \frac{1}{4}$ and $u = \frac{3}{4}$ lie in the Cantor set!

At first glance, this seems paradoxical. We know that $\mathcal{C}$ contains no intervals. Natural intuition from our initial construction therefore suggests that $\mathcal{C}$ contains only the endpoints of deleted open intervals. But such endpoints must have denominator 3^n, so what is $\frac{1}{4}$ doing there?!

The problem isn't the Cantor set, our understanding and intuition with regard to the real numbers is simply weak. Indeed, $\mathcal{C}$ is *uncountably* infinite, whereas there are only *countably* many points in the interval $[0,1]$ with denominator 3^n, so *almost all* points in $\mathcal{C}$ are other than the endpoint of a deleted interval.

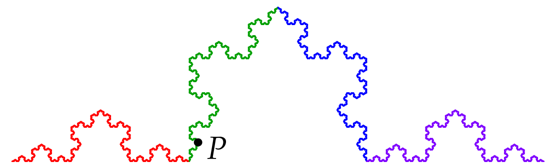
We can repeat the above analysis with other combinations of S_1 and S_2 to uncover more points in Cantor's set.

2. (Example 5.7, cont.) The Koch curve arises from four contraction mappings $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, each with scale factor $c = \frac{1}{3}$ (we use complex numbers for brevity):

Mapping	Effect
$S_1(z) = \frac{1}{3}z$	Scale $\frac{1}{3}$
$S_2(z) = \frac{1}{3}e^{\frac{i\pi}{3}}z + \frac{1}{3}$	Scale $\frac{1}{3}$, rotate 60°, translate
$S_3(z) = \frac{1}{3}e^{\frac{-i\pi}{3}}z + \frac{1}{2} + \frac{\sqrt{3}}{6}i$	Scale $\frac{1}{3}$, rotate −60°, translate
$S_4(z) = \frac{1}{3}z + \frac{2}{3}$	Scale $\frac{1}{3}$, translate

The combined map

$$S(K) := S_1(K) \cup S_2(K) \cup S_3(K) \cup S_4(K)$$

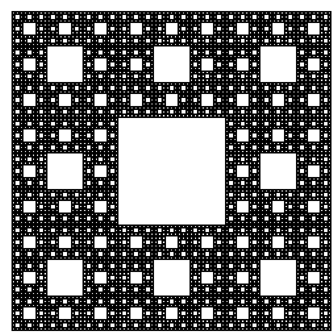


is a contraction on the space $\mathcal{H}$ of non-empty compact subsets $K \subset \mathbb{R}^2$. Regardless of the initial input $K_0 \in \mathcal{H}$, the limit $\lim S^n(K_0)$ is the Koch curve:applied to the entire curve (drawn), the image of each S_i is colored.

We can also play a similar game to the previous example to find interesting points on the curve. For instance, the unique fixed point $P = \left(\frac{51}{146}, \frac{3\sqrt{3}}{146}\right)$ of $U = S_2 \circ S_1$ lies on the curve!

3. The Sierpiński carpet is constructed using eight contraction mappings, each of which scales the entire picture by a (length-scale) factor of $c = \frac{1}{3}$, for a fractal dimension of

$$D = \frac{\log 8}{\log 3} \approx 1.89$$



As with our other examples, the carpet may be realized as the limit of the contraction maps applied to different initial sets K_0.

4. This fractal fern is constructed using three contraction mappings:

   S_1: Scale by $\frac{3}{4}$, rotate 5° clockwise, and translate by $(0, \frac{1}{4})$

   S_2: Scale by $\frac{1}{4}$, rotate 60° counter-clockwise, and translate by $(0, \frac{1}{4})$

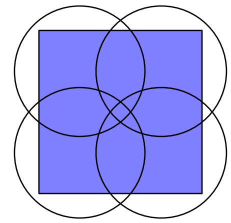
   S_3: Scale by $\frac{1}{4}$, rotate 60° clockwise, and translate by $(0, \frac{1}{4})$

   A linked animation shows the first few steps of the fern's construction starting from a single vertical line segment K_0.

Fractal Dimension Revisited

Since Theorem 5.11 permits several different contraction factors, we need a new approach to fractal dimension. We ask how many disks of a given radius ϵ are required to cover a set.

In the picture, the unit square requires four disks of radius $\varepsilon = 0.4$. For smaller ε, we plainly need more disks...



Definition 5.13. Let K be a compact subset of $\mathbb{R}^m$.

1. If $\varepsilon > 0$, the *closed ε-ball centered at $x \in K$* consists of the points lying at most a distance ε from x:

$$B_\epsilon(x) = \{y \in \mathbb{R}^m : d(x,y) \leq \varepsilon\}$$

2. The *minimal ε-covering number* for K is the smallest number of radius-ε balls needed to cover K:

$$\mathcal{N}(K,\varepsilon) = \min\left\{M : \exists x_1,\ldots,x_M \in K \text{ with } K \subseteq \bigcup_{n=1}^{M} B_\varepsilon(x_n)\right\}$$

3. The *fractal dimension* of K is the limit

$$D = \lim_{\varepsilon\to 0} \frac{\log \mathcal{N}(K,\varepsilon)}{\log(1/\varepsilon)}$$

Rigorously proving that $\mathcal{N}$ and D exist requires a more thorough study of topology, though a simple example should at least convince us that the definition is reasonable.

Example 5.14. Let $K = [0,1]$ be the interval of length 1. It is not hard to check that

$$\varepsilon \geq \frac{1}{2} \iff \mathcal{N}(K,\varepsilon) = 1 \quad \text{and} \quad \frac{1}{4} \leq \varepsilon < \frac{1}{2} \iff \mathcal{N}(K,\varepsilon) = 2$$

etc. More generally, any positive integer $\mathcal{N}$ is related to ϵ via

$$\frac{1}{2\mathcal{N}} \leq \epsilon < \frac{1}{2(\mathcal{N}-1)}$$

The fractal dimension of K $(= 1)$ may be recovered via the squeeze theorem

$$D = \lim_{\varepsilon\to 0} \frac{\log \mathcal{N}}{\log(1/\varepsilon)} = 1$$

The details are left to Exercise 6.

Thankfully, an easier-to-visualize modification is available using boxes.

Theorem 5.15 (Box-counting). *Let $K \subset \mathbb{R}^m$ be compact and cover $\mathbb{R}^m$ by boxes of side length $\frac{1}{2^n}$. Let $\mathcal{N}_n(K)$ be the number of such boxes intersecting K. Then*

$$D = \lim_{n\to\infty} \frac{\log \mathcal{N}_n(K)}{\log 2^n}$$

Our last result is stated without proof: a formula satisfied by the dimension of an iterated function system (Theorem 5.11).

Theorem 5.16. *Let $\{S_n\}_{n=1}^M$ be an iterated function system with attractor (limiting fractal) F and where each contraction S_n has scale factor $c_n \in (0,1)$. Under reasonable conditions,[48] its fractal dimension is the unique real number D satisfying*

$$\sum_{n=1}^{M} c_n^D = 1$$

The existence and uniqueness of D is proved in Exercise 7. When the scale-factors are identical ($c_n = r$), we easily recover Definition 5.4:

$$Mr^D = 1 \implies D = \frac{-\log M}{\log r} = \frac{\log M}{\log(1/r)}$$

Examples 5.17. In more general cases, the Theorem is usually impossible to use *algebraically*; a numerical approximation for D is required.

1. The fractal fern (Examples 5.12.4) is generated by three contraction maps with scale factors $\frac{3}{4}, \frac{1}{4}, \frac{1}{4}$. Its dimension is the solution to the equation
$$\left(\frac{3}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D = 1 \implies D \approx 1.327$$

2. In extreme cases, D may be computed exactly. For instance, if an iterated function system comprises five contraction maps with scale factors $c_1 = c_2 = \frac{1}{2}$ and $c_3 = c_4 = c_5 = \frac{1}{4}$, then the dimension of resulting fractal satisfies
$$2\left(\frac{1}{2}\right)^D + 3\left(\frac{1}{4}\right)^D = 1$$
This is quadratic in $\alpha = \left(\frac{1}{2}\right)^D$, whence
$$2\alpha + 3\alpha^2 = 1 \implies \alpha = \frac{1}{3} \implies D = \log_2 3 \approx 1.58$$

[48] Roughly: the outputs of each S_n applied to F meet only at boundary points; the 'pieces' of the fractal cannot overlap too much.

Other methods of creating fractals

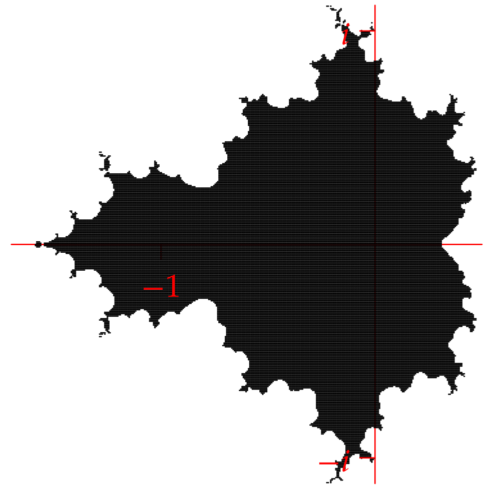
It is hard to give a rigorous definition of *fractal*, though (as with iterated function systems) they typically arise from the repeated application of some function or procedure. Two other famous examples where iterations produce a fractal are the *logistic map* and the *Mandelbrot set*.

The Mandelbrot set (pictured) arises from a construction best described in the complex plane. Given $c \in \mathbb{C}$, iterate the function

$$f_c(z) = z^2 + c$$

If $f\Big(f\big(f(\cdots f(c)\cdots)\big)\Big)$ remains bounded, no matter how many times f is applied, then c lies in the Mandelbrot set.

Far better pictures (and trippy videos) can be found online: sit back and let geometry scramble your conception of reality...



Exercises 5.2. *Key concepts: Contraction mappings, Banach's theorem and its application to fractals, Finding points within fractals, Fractal dimension satisfies* $\sum c_n^D = 1$

1. Let $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ be the contraction mappings defining the Cantor set, and suppose $x, y, z \in \mathbb{R}$ satisfy

$$y = S_1(x), \qquad z = S_2(y), \qquad x = S_2(z)$$

   Show that x, y, z lie in the Cantor set, and find their values.

2. (a) As in Example 5.9, illustrate Banach's Theorem for the contraction mapping

$$S : \mathbb{R} \to \mathbb{R}, \quad S(x) = \frac{1}{2}x + 5$$

   (b) Repeat part (a) for any linear polynomial $S(x) = cx + d$ where $|c| < 1$.

3. Verify the claim in Example 5.12.2 that the point $\left(\frac{51}{146}, \frac{3\sqrt{3}}{146}\right)$ lies on the Koch curve.

4. The construction of a Cantor-type set starts by removing the open intervals $(0.1, 0.2)$ and $(0.6, 0.8)$ from the unit interval.

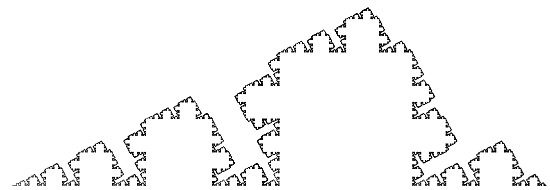
   (a) Sketch the first three iterations of this fractal.

   (b) This construction may be described using three contraction mappings; what are they?

   (c) State an equation satisfied by the dimension D of this fractal and use a computer or calculator to estimate its value.

5. A variation on the Koch curve is constructed using five contraction mappings. Each scales the whole picture by a factor c, then rotates counter-clockwise, before finally translating.

map	scale	rotate	translate (add (x, y))
S_1	$\frac{1}{2}$	0	0
S_2	$\frac{1}{4}$	$90°$	$(\frac{1}{2}, 0)$
S_3	$\frac{1}{4}$	0	$(\frac{1}{2}, \frac{1}{4})$
S_4	$\frac{1}{4}$	$-90°$	$(\frac{3}{4}, \frac{1}{4})$
S_5	$\frac{1}{4}$	0	$(\frac{3}{4}, 0)$



(a) Suppose the initial set K_0 is the straight line segment from $(0, 0)$ to $(1, 0)$. Draw the first two iterations of the fractal's construction.

(b) The dimension of the fractal is the solution D to

$$2^{-D} + 4^{-D} + 4^{-D} + 4^{-D} = 1$$

By observing that $4^{-1} = (2^{-1})^2$, convert this equation to a quadratic in the variable $\alpha = 2^{-D}$. Hence compute the dimension of the fractal.

(c) The dimension of this fractal is *larger* than that of the Koch curve $\left(\frac{\log 4}{\log 3}\right)$. Explain (informally) what this means.

6. Verify the details of Example 5.14, including the computation of the limit.

7. Prove that the value D in Theorem 5.16 exists and is unique.

(*Hint: prove that* $f(x) = \sum c_i^x$ *is strictly decreasing and satisfies the intermediate value theorem...*)

8. (If you've done analysis) Let $S : \mathbb{R} \to \mathbb{R}$ be a contraction mapping with scale factor c, suppose $x_0 \in \mathbb{R}$ is given, and define $x_{n+1} := S(x_n)$ inductively. Prove:

$$\forall k, n \geq 0, \ |x_{n+k} - x_n| < \frac{c^n}{1-c} |x_1 - x_0|$$

Conclude that the sequence (x_n) is Cauchy. Hence prove Banach's Theorem (5.10).