5 Fractal Geometry

5.1 Natural Geometry, Self-similarity and Fractal Dimension

The objects of classical geometry (lines, curves, spheres, etc.) tend to seem flatter and less interesting as one zooms in: at small scales, every differentiable curve looks like a line segment! By contrast, real-world objects tend to exhibit greater detail at smaller scales. A seemingly spherical orange is dimpled on closer inspection: is its surface area that of a sphere, or is the area greater due to the dimples? What if we zoom in further? Under a microscope, the dimples in the orange are seen to have minute cracks and fissures. With modern technology, we can 'see' almost to the molecular level; what does *surface area* even mean at such a scale?

The Length of a Coastline In 1967 Benoit Mandelbrot asked a related question in a now-famous paper, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension.* His essential point was that this question has no simple answer:[34] Should one measure by walking along the mean high tide line? But where is this? Do we 'walk' round every pebble? Do we skirt every grain of sand? Every molecule? As the scale of consideration shrinks, the measured length becomes absurdly large. Here is a sketch of Mandelbrot's approach.[35]

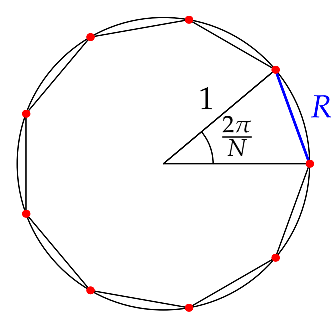
- Given a ruler of length R, let N be the number required to trace round the coastline when laid end-to-end.
- Plot $\log N$ against $\log(1/R)$ for several sizes of ruler. The data suggests a straight line!

$$\log N \approx \log k + D\log(1/R) = \log(kR^{-D}) \implies N \approx kR^{-D}$$

The number D is Mandelbrot's *fractal dimension* of the coastline.

This notion of fractal dimension is purely empirical, though it does seem to capture something about the 'roughness' of a coastline: the bumpier the coast, the greater its fractal dimension. For mainland Britain with its smooth east and rugged west coasts $D \approx 1.25$. Given its many fjords, Norway has a far rougher coastline and thus a higher fractal dimension $D \approx 1.52$.

Example 5.1. As a sanity check, consider a smooth circular 'coastline.' Approximate the circumference using N rulers of length R: clearly



$$R = 2\sin\frac{\pi}{N}$$

As $N \to \infty$, the small angle approximation for sine applies,

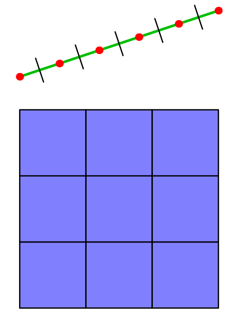
$$R \approx \frac{2\pi}{N} \implies N \approx 2\pi R^{-1}$$

where the approximation improves as $N \to \infty$. The fractal dimension of a circle is therefore $D = 1$. The same analysis applies to any smooth curve (Exercise 3).

[34]The official answer from the Ordnance Survey (the UK government mapping office) is, 'It depends.' The all-knowing CIA states 7723 miles, though offers no evidence as to why.

[35]For more detail see the Fractal Foundation's website. Mandelbrot coined the word *fractal*, though he didn't invent the concept from nothing. Rather he applied earlier ideas of Hausdorff, Minkowski and others, and observed how the natural world contains many examples of fractal structures.

Our goal is to describe a related notion of fractional dimension for *self-similar* objects. To help motivate the definition, recall some of the standard objects of pre-fractal geometry.



Segment A segment can be viewed as N copies of itself each scaled by a factor $r = \frac{1}{N}$.

Square A square comprises N copies of itself scaled by a factor $r = \frac{1}{\sqrt{N}}$.

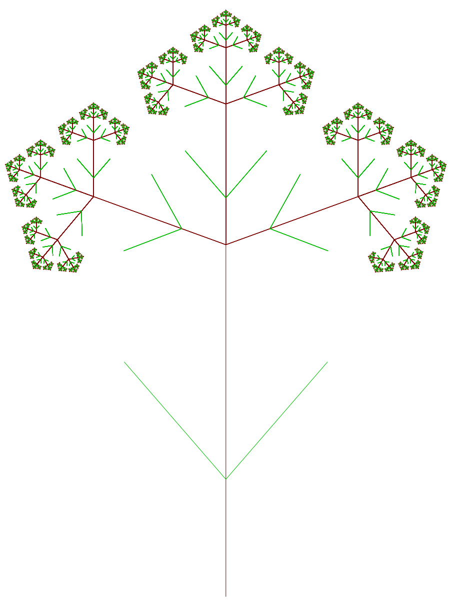
Cube A cube comprises N copies of itself scaled by a factor $r = \frac{1}{\sqrt[3]{N}}$.

In each case observe that $N = \left(\frac{1}{r}\right)^D$ where D is the usual dimension of the object (1, 2 or 3). Inspired by this, we make a loose definition.

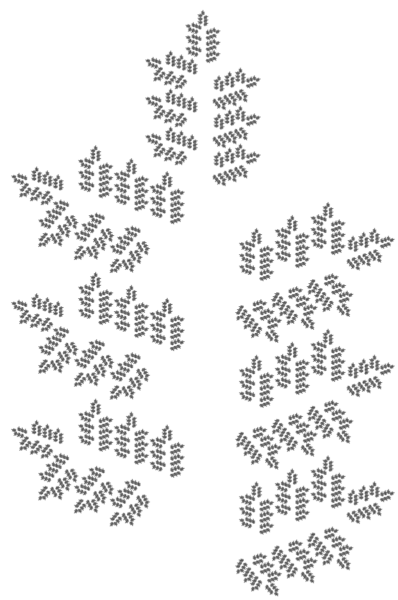
Definition 5.2. A geometric figure is *self-similar* if it may be subdivided into N similar copies of itself, each scaled by a magnification factor $r < 1$. The *fractal dimension* of such a figure is

$$D := \log_{1/r} N = \frac{\log N}{\log(1/r)} = -\frac{\log N}{\log r}$$

Example 5.3. The botanical pictures below offer some evidence for non-integer fractal dimension and for the idea that self-similarity is a natural phenomenon. The 'tree' comprises $N = 3$ copies of itself, each scaled by $r = 0.4$. Its fractal dimension is therefore $D = -\frac{\log 3}{\log 0.4} \approx 1.199$.
The fern has $N = 7$ and $r = 0.3$ for a fractal dimension $D = -\frac{\log 7}{\log 0.3} \approx 1.616$.



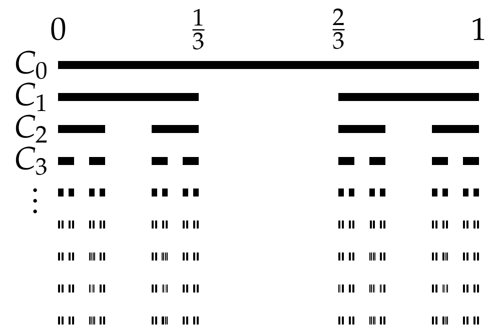
Tree fractal $D \approx 1.199$



Fern fractal $D \approx 1.616$

With dimensions between 1 and 2, both objects exhibit an intuitive idea of fractal dimension: both seem to occupy more space than mere *lines*, but neither has positive *area*. Moreover, the fern seems to occupy more space—has higher dimension—than the tree. (The 'trunk' and 'branches' in the first picture aren't really part of the fractal and are drawn only to give the picture a skeleton.)

Example 5.4 (Cantor's Middle-third Set). This famous example dates from the late 1800s.[36]



Starting with the unit interval $C_0 = [0, 1]$, define a sequence of sets (C_n) where C_{n+1} is obtained by deleting the open 'middle-third' of each interval in C_n; for instance

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Cantor's set is essentially the limit of this sequence:

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n$$

Cantor's set has several strange properties, none of which we establish rigorously.

Zero length The sum of the lengths of the disjoint sub-intervals comprising C_n is $\text{length}(C_n) = \left(\frac{2}{3}\right)^n$ since we delete $\frac{1}{3}$ of the remaining set at each step. It follows that

$$\forall n \in \mathbb{N}_0, \ \text{length}(\mathcal{C}) \le \left(\frac{2}{3}\right)^n \implies \text{length}(\mathcal{C}) = 0$$

We conclude that $\mathcal{C}$ contains no subintervals!

Uncountability There exists a bijection between $\mathcal{C}$ and the original interval $[0, 1]$! (This issue is of limited interest to us, though you've likely encountered the notion elsewhere.)

Self-similarity Since C_{n+1} consists of two copies of C_n, each shrunk by a factor of $\frac{1}{3}$ and one shifted $\frac{2}{3}$ to the right, we abuse notation slightly to write

$$C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{1}{3}C_n + \frac{2}{3}\right)$$

'Taking limits,' Cantor's set is seen to comprise two shrunken copies of itself:

$$\mathcal{C} = \frac{1}{3}\mathcal{C} \cup \left(\frac{1}{3}\mathcal{C} + \frac{2}{3}\right)$$

In particular, its fractal dimension is $D = \frac{\log 2}{\log 3} \approx 0.631$.

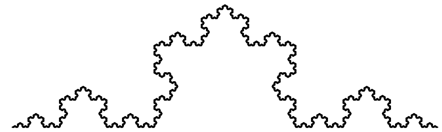
The Cantor set has many generalizations:

- Removing different fractions of every interval at each stage produces sets with other fractal dimensions. For instance, removing the 2nd and 4th fifths results in $D = \frac{\log 3}{\log 5} \approx 0.683$.
- Higher-dimensional analogues include the Sierpiński triangle ($D = \frac{\log 3}{\log 2} \approx 1.585$) and carpet (Example 5.10.3, $D = \frac{\log 8}{\log 3} \approx 1.893$), and the Menger sponge ($D = \frac{\log 20}{\log 3} \approx 2.727$).

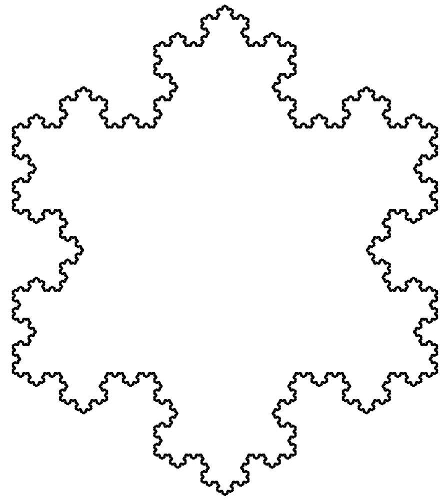
[36] Henry Smith discovered this set in 1874 while investigating integrability (the 'length' of a set was later formalized using *measure theory*). Cantor's 1883 description focused on topological properties, with self-similarity being less of a concern.

Example 5.5 (The Koch Curve and Snowflake). Another generalization of the Cantor set is produced as the limit of a sequence of curves.

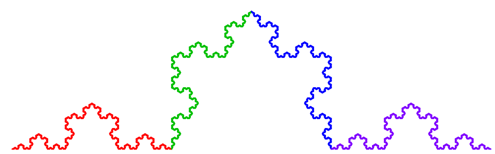
- Let K_0 be a segment of length 1.
- Replace the middle third of K_0 with the other two sides of an equilateral triangle to create K_1.
- Replace the middle third of each segment in K_1 as before to create K_2.
- Repeat *ad infinitum.*



Koch Curve



Koch Snowflake



Self-similarity

The resulting curve is drawn along with the *Koch snowflake* obtained by arranging three copies around an equilateral triangle.

The relation to the Cantor set should be obvious in the construction. Indeed if $K_0 = [0,1]$, then the intersection of this with the Koch curve is the Cantor set itself!

The Koch curve is self-similar in that it comprises $N = 4$ copies of itself shrunk by a factor of $r = \frac{1}{3}$. Its fractal dimension is therefore $\frac{\log 4}{\log 3} \approx 1.2619$, between that of a line and an area.

We may also consider the curve's length. Let s_n be the number of segments in K_n, each having length t_n, and let $\ell_n = t_n s_n$ be the length of the curve K_n. It follows that

$$s_n = 4^n, \quad t_n = \frac{1}{3^n} \implies \ell_n = \left(\frac{4}{3}\right)^n \to \infty$$

The Koch curve is *infinitely long*!

Exercises 5.1. 1. By removing a constant middle fraction of each interval, construct a fractal analogous to the Cantor set but with dimension $\frac{1}{2}$. More generally, if one removes a constant middle fraction f from each interval, what is the resulting dimension?

2. Prove that the area inside the n^{th} iteration of the construction of the Koch snowflake is

$$A_n = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\left[1 - \left(\frac{4}{9}\right)^n\right]\right) \xrightarrow[n\to\infty]{} \frac{2\sqrt{3}}{5} = \frac{8}{5}\,\text{Area}(\triangle)$$

3. Suppose $\mathbf{r}(t)$, $t \in [0,1]$ describes a regular (smooth) curve in the plane.

   (a) Use the arc-length formula $L = \int_0^1 |\mathbf{r}'(t)|\,\mathrm{d}t$ together with Riemann sums and the linear approximation $\mathbf{r}(t + \frac{1}{N}) \approx \mathbf{r}(t) + \frac{1}{N}\mathbf{r}'(t)$ to argue that

   $$L \approx \sum_{k=0}^{N-1} \left| \mathbf{r}\left(\frac{k+1}{N}\right) - \mathbf{r}\left(\frac{k}{N}\right) \right| \tag{$*$}$$

   (b) Parametrizing $\mathbf{r}$ such that each segment in $(*)$ has the same length R, prove that $L \approx NR$. (*Any regular curve thus has fractal dimension 1 in the sense stated by Mandelbrot (pg. 81)*)

5.2 Contraction Mappings & Iterated Function Systems

Thus far we have dealt informally with fractals where the whole consists of multiple pieces scaled by the same factor. In general we can mix up scaling factors. To do this, and to be more rigorous, we need to borrow some ideas from topology and analysis.

Definition 5.6. A *contraction mapping* with scale factor $c \in [0,1)$ is a function $S : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\forall x, y \in \mathbb{R}^m, \quad |S(x) - S(y)| \le c\,|x-y|$$

A contraction mapping moves inputs closer together. It should be clear that every such is continuous ($\lim x_n = y \implies \lim S(x_n) = S(y)$).

Example 5.7. The function $S(x) = \frac{1}{3}x + \frac{2}{3}$ is a contraction mapping (on $\mathbb{R}$) with scale factor $c = \frac{1}{3}$:

$$|S(x) - S(y)| = \frac{1}{3}\,|x-y|$$

To motivate the key theorem, consider using S to inductively define a sequence: given any x_0, define

$$x_{n+1} := S(x_n)$$

The first few terms are

$$x_1 = \frac{x_0}{3} + \frac{2}{3}, \qquad x_2 = \frac{x_0}{3^2} + \frac{2}{3^2} + \frac{2}{3}, \qquad x_3 = \frac{x_0}{3^3} + \frac{2}{3^3} + \frac{2}{3^2} + \frac{2}{3}$$

which suggests (geometric series)

$$x_n = \frac{x_0}{3^n} + 2\sum_{k=1}^{n} \frac{1}{3^k} = \frac{x_0}{3^n} + \frac{2(3^{-1} - 3^{-n-1})}{1 - 3^{-1}} = 1 + \frac{1}{3^n}(x_0 - 1)$$

This can easily be verified by induction if you prefer. The striking thing about this sequence is that it converges to the same limit $\lim x_n = 1$ *regardless of the initial term* x_0!

The example illustrates one of the most powerful and useful theorems in mathematics.

Theorem 5.8 (Banach Fixed Point Theorem). *Let $S : \mathbb{R} \to \mathbb{R}$ be a contraction mapping. Then:*

1. *S has a unique **fixed point**: some $L \in \mathbb{R}^m$ such that $S(L) = L$.*
2. *If $x_0 \in \mathbb{R}$ is **any** value, then the sequence defined iteratively by $x_{n+1} := S(x_n)$ converges to L.*

In fact, as will be crucial momentarily, Banach's result holds whenever $S : \mathcal{H} \to \mathcal{H}$ is a contraction mapping on any *complete metric space*.[37] The main goal of this section is to use Banach's result to generate certain fractals via repeated application of contraction mappings to an initial shape. Our motivating example already illustrates this…

[37] Very loosely, a metric space is a set on which a sensible notion of *distance* can be defined: on $\mathbb{R}$, for instance, the distance between two points is $d(x,y) = |x-y|$. If you've done analysis you'll be familiar with *completeness*: every Cauchy sequence converges (in $\mathcal{H}$).

Example (5.4, Cantor Set mk. II). The functions $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ where

$$S_1(x) = \frac{x}{3} \qquad S_2(x) = \frac{x}{3} + \frac{2}{3}$$

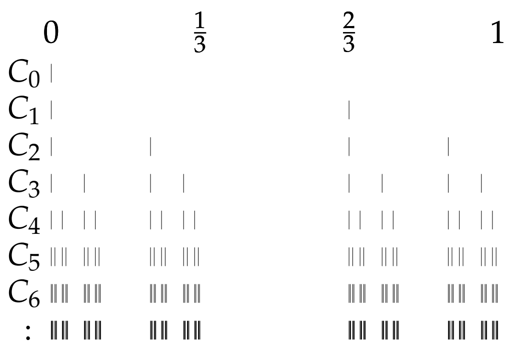
are contraction mappings with scale factor $c = \frac{1}{3}$. More importantly, these functions *define* the Cantor set: at each stage of its construction, we defined

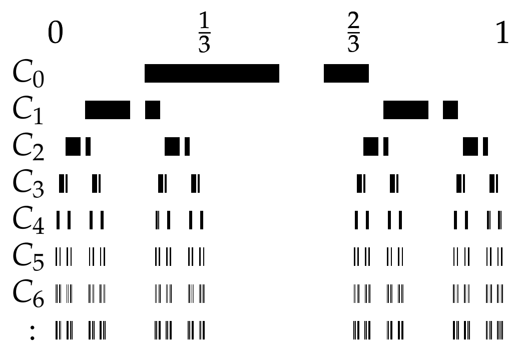
$$C_{n+1} := S_1(C_n) \cup S_2(C_n) \tag{$*$}$$

Indeed, the self-similarity of the Cantor set can be expressed in the same manner: $\mathcal{C} = S_1(\mathcal{C}) \cup S_2(\mathcal{C})$. Amazingly, it barely seems to matter what initial set C_0 is chosen. Originally we took $C_0 = [0, 1]$ to be the unit interval, but we could instead start with the singleton set $C_0 = \{0\}$: iterating $(*)$ produces

$$C_1 = \{0, \tfrac{2}{3}\}, \qquad C_2 = \{0, \tfrac{2}{9}, \tfrac{2}{3}, \tfrac{8}{9}\}, \qquad C_3 = \{0, \tfrac{2}{27}, \tfrac{2}{9}, \tfrac{8}{27}, \tfrac{2}{3}, \tfrac{20}{27}, \tfrac{8}{9}, \tfrac{26}{27}\}, \ \ldots$$

The first few iterations are drawn in the first picture below; it appears as if, in the limit, C_n is becoming the Cantor set. The second picture starts with a very different initial set $C_0 = [0.2, 0.5] \cup [0.6, 0.7]$; iterating this also appears to produce the Cantor set!





It certainly appears as if the Cantor set is generated by the contraction maps S_1, S_2 independently of the initial data C_0. Our main result shows in what sense this is true. Since this requires some heavy lifting from topology and analysis, we provide only a synopsis.

- A subset $K \subset \mathbb{R}^m$ is *compact* if it is *closed* (contains its boundary points) and *bounded* (K lies within some ball centered at the origin). For instance, $K = [0, 1]$ is a compact subset of $\mathbb{R}$.
- The set of non-empty compact subsets of $\mathbb{R}^m$ is a *metric space* $\mathcal{H}$. This means that the *distance* $d(X, Y)$ between $X, Y \in \mathcal{H}$ may sensibly be defined, though it is a little tricky…[38]

[38] The distance function is the *Hausdorff metric*. Given $Y \in \mathcal{H}$, and $x \in \mathbb{R}^n$, define $d_Y(x) = \inf_{y \in Y} ||x - y||$ to be the distance from x to the 'nearest' point of Y. Define $d_X(y)$ similarly. The Hausdorff distance between X and Y is then

$$d(X, Y) := \max\left\{\sup_{x \in X} d_Y(x), \sup_{y \in Y} d_X(y)\right\}$$

Roughly speaking, find $x \in X$ which is as far away ($d_Y(x)$) as possible from anything in Y, and find $y \in Y$ similarly; $d(X, Y)$ is the larger of these distances. Crucially, $d(X, Y) = 0 \iff X = Y$.

- Since $\mathcal{H}$ is a metric space, we can discuss convergent sequences (K_n) of compact sets

  $$\lim_{n\to\infty} K_n = K \iff \lim_{n\to\infty} d(K_n, K) = 0$$

  It also makes sense to speak of Cauchy sequences in $\mathcal{H}$. It may be proved that $\mathcal{H}$ is *complete*: every Cauchy sequence $(K_n) \subseteq \mathcal{H}$ converges to some $K \in \mathcal{H}$.

- The main result is a corollary of Banach's result (Theorem 5.8).

Theorem 5.9 (Iterated Function Systems). *Let $S_1, \ldots, S_n$ be contraction mappings on $\mathbb{R}^m$ with scale factors $c_1, \ldots, c_n$. Define*

$$S : \mathcal{H} \to \mathcal{H} \quad \text{by} \quad S(K) = \bigcup_{i=1}^{n} S_i(K)$$

1. *S is a contraction mapping on $\mathcal{H}$ with contraction factor $c = \max(c_1, \ldots, c_n)$.*
2. *S has a unique fixed set $F \in \mathcal{H}$ given by $F = \lim_{k\to\infty} S^k(K_0)$ for **any** non-empty $K_0 \in \mathcal{H}$.*

Part 1 is not super difficult to prove if you're willing to work with the definition of the Hausdorff metric (try it if you're comfortable with analysis!). Part 2 is Banach's theorem.

The upshot is this: repeatedly applying contraction mappings to *any* non-empty compact set E produces a compact limit set which is *independent of E!* We call the limit F for *fractal.* Such fractals are sometimes called *attractors*: being limit-sets, they 'attract' data towards themselves.

Examples 5.10. 1. (Example 5.4, mk.III) We revisit the Cantor set one last time.

The contractions $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ (on $\mathbb{R}$) produce a contraction $S : \mathcal{H} \to \mathcal{H}$:

$$S(K) := \{S_1(x), S_2(x) : x \in K\}$$

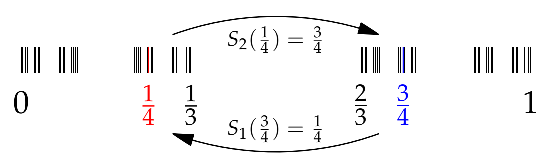
By Theorem 5.9, if $C_0 \subset \mathbb{R}$ is non-empty closed and bounded, then $\mathcal{C} = \lim S^n(C_0)$. Certainly all three of our previous choices for C_0 are such sets: $[0,1]$, $\{0\}$ and $[0.2, 0.5] \cup [0.6, 0.7]$.

A nice application of the Theorem allows us to find all sorts of interesting points in the Cantor set. For instance, consider the functions T, U, where

$$T(x) = S_1\big(S_2(x)\big) = \frac{x}{9} + \frac{2}{9} \quad \text{and} \quad U(y) = S_2\big(S_1(y)\big) = \frac{y}{9} + \frac{2}{3}$$

These are contractions on $\mathbb{R}$ ($c = \frac{1}{9}$) whose unique fixed points are $t = \frac{1}{4}$ and $u = \frac{3}{4}$; moreover $S_1(u) = t$ and $S_2(t) = u$. Now consider the non-empty compact set $K = \{t, u\} \in \mathcal{H}$. Plainly

$$S(K) = \{S_1(t), S_1(u), S_2(t), S_2(u)\} = \left\{\frac{1}{12}, \frac{1}{4}, \frac{3}{4}, \frac{11}{12}\right\} \supset K$$



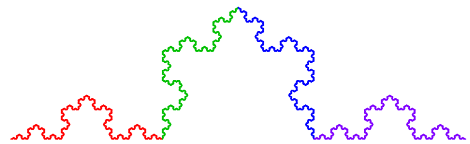
It follows (induction) that $K \subseteq \lim S^n(K) = \mathcal{C}$: both $t = \frac{1}{4}$ and $u = \frac{3}{4}$ lie in the Cantor set! This seems paradoxical: $\frac{1}{4}$ does not lie at the end of any deleted interval (all such points have denominator 3^n) but yet the Cantor set contains no intervals. How does $\frac{1}{4}$ end up in there?!

2. (Example 5.5) The Koch curve arises from four contraction mappings $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, each with scale factor $c = \frac{1}{3}$.

Mapping	Effect
$S_1(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right)$	Scale $\frac{1}{3}$
$S_2(x,y) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right)$	Scale $\frac{1}{3}$, rotate 60°, translate
$S_3(x,y) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, \frac{\sqrt{3}}{6}x - \frac{1}{6}y + \frac{\sqrt{3}}{6}\right)$	Scale $\frac{1}{3}$, rotate $-60°$, translate
$S_4(x,y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right)$	Scale $\frac{1}{3}$, translate

The combined map

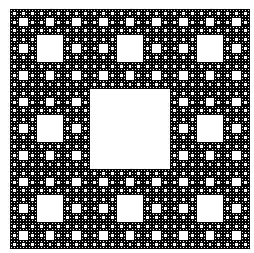
$$S(K) := S_1(K) \cup S_2(K) \cup S_3(K) \cup S_4(K)$$



is a contraction on $\mathcal{H} = \{\text{non-empty compact } K \subset \mathbb{R}^2\}$.
Regardless of the initial input $K_0 \in \mathcal{H}$, the limit $\lim S^n(K_0)$ is the Koch curve: applied to the entire curve (drawn), the image of each S_i is colored. The picture moreover links to a series of animated constructions starting with different initial sets K_0.

We can also play a similar game to the previous example to find interesting points on the curve. For instance, the unique fixed point $\left(\frac{51}{146}, \frac{3\sqrt{3}}{146}\right)$ of $U = S_2 \circ S_1$ lies on the curve!

3. The Sierpiński carpet may be constructed using eight contraction mappings, each of which scales the whole picture by a (length-scale) factor of $c = \frac{1}{3}$, for a dimension of $D = \frac{\log 8}{\log 3} \approx 1.893$.
As with the Koch curve, the image links to several alternative constructions using different initial sets K_0.



4. This fractal fern is constructed using three contraction mappings:

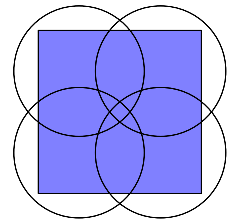
   S_1: Scale by $\frac{3}{4}$, rotate 5° clockwise, and translate by $(0, \frac{1}{4})$

   S_2: Scale by $\frac{1}{4}$, rotate 60° counter-clockwise, and translate by $(0, \frac{1}{4})$

   S_3: Scale by $\frac{1}{4}$, rotate 60° clockwise, and translate by $(0, \frac{1}{4})$

   The linked animation shows the first few steps of its contsruction starting from a single vertical line segment K_0.

Fractal Dimension Revisited



Since Theorem 5.9 permits several different contraction factors, we need a new approach to fractal dimension. We ask how many disks of a given radius ϵ are required to cover a set. In the picture, the unit square requires four disks of radius $\varepsilon = 0.4$. For smaller ε, we plainly need more disks...

Definition 5.11. Let K be a compact subset of $\mathbb{R}^m$.

1. If $\varepsilon > 0$, the *closed ε-ball centered at $x \in K$* consists of the points at most a distance ε from x:

$$B_\epsilon(x) = \{y \in \mathbb{R}^m : d(x,y) \le \varepsilon\}$$

2. The *minimal ε-covering number* for K is the smallest number of radius-ϵ balls needed to cover K:

$$\mathcal{N}(K,\varepsilon) = \min\left\{ M : \exists x_1,\ldots,x_M \in K \text{ with } K \subseteq \bigcup_{n=1}^{M} B_\epsilon(x_n) \right\}$$

3. The *fractal dimension* of K is the limit

$$D = \lim_{\varepsilon \to 0} \frac{\log \mathcal{N}(K,\varepsilon)}{\log(1/\varepsilon)}$$

Rigorously proving that $\mathcal{N}$ and D exist requires a more thorough study of topology, though a simple example should at least convince us that the definition is reasonable!

Example 5.12. Let $K = [0,1]$ be the interval of length 1. It is not hard to check that

$$\varepsilon \ge \frac{1}{2} \iff \mathcal{N}(K,\varepsilon) = 1 \quad \text{and} \quad \frac{1}{4} \le \varepsilon < \frac{1}{2} \iff \mathcal{N}(K,\varepsilon) = 2$$

etc. More generally, $\mathcal{N}$ and ϵ are related via

$$\frac{1}{2\mathcal{N}} \le \epsilon < \frac{1}{2(\mathcal{N}-1)}$$

The dimension of K ($= 1$) may therefore be recovered via the squeeze theorem

$$D = \lim_{\epsilon \to 0} \frac{\log \mathcal{N}}{\log(1/\varepsilon)} = 1$$

Thankfully an easier-to-visualize modification is available using boxes.

Theorem 5.13 (Box-counting). *Let $K \subset \mathbb{R}^m$ be compact and cover $\mathbb{R}^m$ by boxes of side length $\frac{1}{2^n}$. Let $\mathcal{N}_n(K)$ be the number of such boxes intersecting K. Then*

$$D = \lim_{n \to \infty} \frac{\log \mathcal{N}_n(K)}{\log 2^n}$$

We finish with a formula satisfied by the dimension of an iterated function system (Theorem 5.9).

Theorem 5.14. *Let $\{S_n\}_{n=1}^{M}$ be an iterated function system with attractor (limiting fractal) F and where each contraction S_n has scale factor $c_n \in (0,1)$. Under reasonable conditions,*[39] *the fractal dimension is the unique real number D satisfying*

$$\sum_{n=1}^{M} c_n^D = 1$$

Examples 5.15. 1. We easily recover Definition 5.2 when the scale-factors are identical $c_n = r$:

$$Mr^D = 1 \implies D = \frac{-\log M}{\log r} = \frac{\log M}{\log(1/r)}$$

2. The fractal fern (Examples 5.10) is generated by three contraction maps with scale factors $\frac{3}{4}, \frac{1}{4}, \frac{1}{4}$. Its dimension is the solution to the equation

$$\left(\frac{3}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D = 1 \implies D \approx 1.3267$$

3. Numerical approximation is usually required to find D, though sometimes an exact solution is possible. For instance, if $c_1 = c_2 = \frac{1}{2}$ and $c_3 = c_4 = c_5 = \frac{1}{4}$, then

$$2\left(\frac{1}{2}\right)^D + 3\left(\frac{1}{4}\right)^D = 1$$

This is quadratic in $\alpha = \left(\frac{1}{2}\right)^D$, whence

$$2\alpha + 3\alpha^2 = 1 \implies \alpha = \frac{1}{3} \implies D = \log_2 3 \approx 1.584$$

Other methods of creating fractals

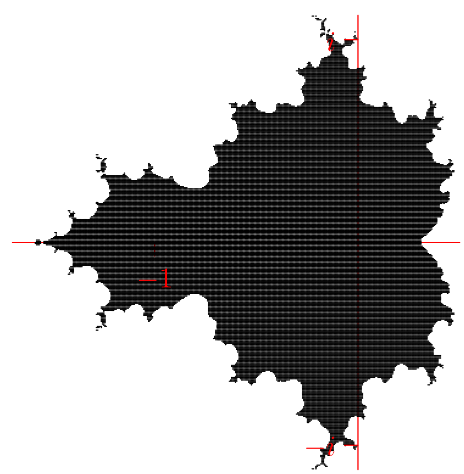
The contraction mapping approach is one of many ways to create fractals. Two other famous examples are the *logistic map* (related to numerical approximations to non-linear differential equations) and the *Mandelbrot set* (pictured).

The Mandelbrot set arises from a construction in the complex plane. For a given $c \in \mathbb{C}$, we iterate the function

$$f_c(z) = z^2 + c$$

If $f(f(f(\cdots f(c)\cdots)))$ remains bounded, no matter how many times f is applied, then c lies in the Mandelbrot set.

Much better pictures and trippy videos can be found online...



[39] Roughly: the outputs of each S_n meet only at boundary points; the 'pieces' of the fractal cannot overlap too much.

Exercises 5.2. 1. Let $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ be the contraction mappings defining the Cantor set and suppose $x, y, z \in \mathbb{R}$ satisfy

$$y = S_1(x), \qquad z = S_2(y), \qquad x = S_2(z)$$

Show that x, y, z lie in the Cantor set, and find their values.

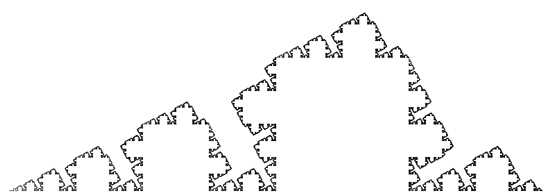
2. (a) As in Example 5.7, illustrate Banach's theorem for the contraction $S(x) = \frac{1}{2}x + 5$.
   (b) Repeat part (a) for any linear polynomial $S(x) = cx + d$ where $|c| < 1$.

3. Verify the claim in Example 5.10.2 that the point $(\frac{51}{146}, \frac{3\sqrt{3}}{146})$ lies on the Koch curve.

4. The construction of a Cantor-type set starts by removing the open intervals $(0.1, 0.2)$ and $(0.6, 0.8)$ from the unit interval.
   (a) Sketch the first three iterations of this fractal.
   (b) This construction may be described using three contraction mappings; what are they?
   (c) State an equation satisfied by the dimension D of the set and use a computer algebra package to estimate its value.

5. A variation on the Koch curve is constructed using five contraction mappings. Each scales the whole picture by a factor c, then rotates counter-clockwise, before finally translating.

map	scale	rotate	translate (add (x, y))
S_1	$\frac{1}{2}$	0	0
S_2	$\frac{1}{4}$	90°	$(\frac{1}{2}, 0)$
S_3	$\frac{1}{4}$	0	$(\frac{1}{2}, \frac{1}{4})$
S_4	$\frac{1}{4}$	−90°	$(\frac{3}{4}, \frac{1}{4})$
S_5	$\frac{1}{4}$	0	$(\frac{3}{4}, 0)$



   (a) Suppose your initial set K_0 is the straight line segment from $(0, 0)$ to $(1, 0)$. Draw the first two iterations of the fractal's construction.
   (b) The dimension of the fractal is the solution D to $(\frac{1}{2})^D + (\frac{1}{4})^D + (\frac{1}{4})^D + (\frac{1}{4})^D + (\frac{1}{4})^D = 1$. By observing that $\frac{1}{4} = \left(\frac{1}{2}\right)^2$, convert to a quadratic equation in the variable $\alpha = (\frac{1}{2})^D$. Hence compute the dimension of the fractal.
   (c) The dimension of the fractal is *larger* than that of the Koch curve $\left(\frac{\log 4}{\log 3}\right)$. Explain (informally) what this means.

6. Verify the details of Example 5.12, including the computation of the limit.

7. Given constants $0 \le c_1, \ldots, c_n < 1$, use calculus to prove that the function $f(x) = \sum c_i^x$ is strictly decreasing. Hence conclude that the value D in Theorem 5.14 exists and is unique.

8. (If you've done analysis) Let $S : \mathbb{R} \to \mathbb{R}$ be a contraction mapping with scale factor c, suppose $x_0 \in \mathbb{R}$ is given, and define $x_{n+1} := S(x_n)$ inductively. Prove:

$$\forall k, n \ge 0, \; |x_{n+k} - x_n| < \frac{c^n}{1-c} |x_1 - x_0|$$

Conclude that the sequence (x_n) is Cauchy. Hence prove Banach's Theorem (5.8).