Math 161: Homework 1

Submit your answers to the following questions at the discussion class on Thursday 18th January

1. In the game of Nim, two players are given several piles of coins, each pile having a finite number of coins. On each turn a player picks a pile and removes as many coins as they want from that pile, as long as they remove at least one coin. The player who takes the last coin wins. Suppose that there are two piles, where one pile has more coins than the other. Prove that the first player to move can always win the game.

2. Consider a system where we have children in a classroom choosing different flavors of ice cream. Suppose we have the following axioms:

   A1 There are exactly five flavors of ice cream: vanilla, chocolate, strawberry, cookie dough, and bubble gum.
   A2 Given any two different flavors, there is exactly one child who likes these two flavors.
   A3 Every child likes exactly two flavors of ice cream.

   (a) How many children are in the classroom? Prove your assertion.
   (b) Prove that any pair of children likes at most one common flavor.
   (c) Prove that for each flavor, there are exactly four children who like that flavor.

3. Consider an axiomatic system that consists of elements in a set $S$ and a set $P$ of pairings of elements $(a,b)$ that satisfy the following axioms:

   A1 If $(a,b)$ is in $P$, then $(b,a)$ is not in $P$.
   A2 If $(a,b)$ is in $P$ and $(b,c)$ is in $P$, then $(a,c)$ is in $P$.

   (a) Let $S = \{1,2,3,4\}$ and $P = \{(1,2), (2,3), (1,3)\}$. Is this a model for the axiomatic system? Why/why not?
   (b) Now let $S$ be the set of real numbers and let $P$ consist of all pairs $(x,y)$ where $x < y$. Is this a model for the system? Explain.
   (c) Finally, use the results of (a) and (b) to argue that the axiomatic system with sets $S$ and $P$ is not complete. I.e., think of another independent axiom that could be added to the axioms A1 and A2 for which $S$ and $P$ in part (a) is a model, but for which $S$ and $P$ from part (b) is not a model.

4. Show how the algebraic identity $(a + b)^2 = a^2 + 2ab + b^2$ can be established geometrically. Can you make your argument work even if $a$ or $b$ are $\leq 0$?