Math 161: Homework Questions 3

Submit your answers to the * questions at the discussion class on Thursday 31st January

1. * Let \( \ell \) be a line and \( A, B, C \) be three distinct points. Suppose:
   - \( A \) and \( B \) are on the same side of \( \ell \).
   - \( B \) and \( C \) are on the same side of \( \ell \).

   Use Pasch’s Axiom to prove that \( A \) and \( C \) lie on the same side of \( \ell \).

2. (a) Give a sensible definition in terms of Hilbert’s system of the interior of a triangle \( \triangle ABC \).
   (b) Prove that the interior of any triangle is non-empty.
   (Hint: recall the fact that between any two points there exists a third . . .)

3. * We prove Euclid’s Theorems I.18, 19 and 20, on comparisons of angles and sides in a triangle. Suppose we have \( \triangle ABC \): for convenience we use angle and length measure and write

   \[ a = |BC|, \quad b = |AC|, \quad c = |AB|, \]
   \[ \alpha = m\angle CAB, \quad \beta = m\angle ABC, \quad \gamma = m\angle BCA \]

   (a) (I.18) Assume \( a < c \). Construct \( D \) on \( AB \) such that \( |BD| = a \). Prove that \( \alpha < \gamma \).

   (b) (I.19) Prove the converse: if \( \alpha < \gamma \), then \( a < c \).
   (Hint: Show that \( a \geq c \) is impossible . . .)

   (c) (I.20) Prove the triangle-inequality. For any triangle, \( a + b > c \).
   (Hint: Let \( E \) lie on \( BC \) such that \( |CE| = b \): try to apply part (b))

   (If you want more of a challenge, re-write the questions and all your arguments without mentioning length or angle measure.)

4. Prove the SAS congruence theorem for similar triangles: to make things simpler, use the set-up in the picture and prove.

   \[ \triangle ABC \sim \triangle ADE \iff \frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} \]

   (Hint: construct \( BF \) parallel to \( DE \) and appeal to AAA . . .)
5. (a) A median of a triangle is a segment from the midpoint of a side to the opposite vertex. Use Ceva’s Theorem to prove that the three medians of a triangle meet at a point (the centroid).
(b) (Hard) The medians of a triangle split the triangle into six sub-triangles. Prove that all six sub-triangles have the same area.
(c) Prove that the centroid is exactly 2/3 of the distance along each median.

6. * Let $\triangle ABC$ have a right-angle at $C$. Drop a perpendicular from $C$ to $\overline{AB}$ at $D$.
   
   (a) Prove that $D$ lies on $\overline{AB}$.
   (b) Prove that you have three similar triangles.
   (c) Use these facts to prove Pythagoras’ Theorem.

7. * Let $\overline{AD}$ and $\overline{BC}$ be two chords of a circle that intersect at $P$. Show that $|AP| \cdot |BP| = |PC| \cdot |PD|$. (Hint: use similar triangles.)

8. Give proofs of all three pieces of Corollary 2.33 from the notes.

9. * Justify the assertion at the end of Theorem 2.37: why are the intersection points of the given circles also the required points of tangency?

10. * Two circles meet at points $P$ and $Q$. Let $\overline{AP}$ and $\overline{BP}$ be diameters of the circles. Prove that $\overline{AB}$ passes through the other intersection point $Q$.

11. Let an angle $\alpha$ be less than a straight edge. We say $\alpha$ is acute if it is less than a right-angle, and obtuse if it is greater. Its supplementary angle $\hat{\alpha}$ is the angle such that $\alpha + \hat{\alpha}$ is a straight edge.
   
   (a) Prove that all triangles fall into exactly one of three cases:
      1. The triangle is right-angled.
      2. The triangle has three acute angles.
      3. The triangle has one obtuse and two acute angles.
      Your answer should not mention any measure (degrees, radians) of angles.
   (b) In class we defined the sine and cosine of an acute angle $\alpha$ as ratios of lengths. Extend the definitions to right- and obtuse angles as follows:
      \[
      \sin \alpha = \begin{cases} 
      1 & \text{if } \alpha \text{ is a right angle} \\
      \sin \hat{\alpha} & \text{if } \alpha \text{ is acute} \\
      \end{cases}
      \]
      \[
      \cos \alpha = \begin{cases} 
      0 & \text{if } \alpha \text{ is a right angle} \\
      -\cos \hat{\alpha} & \text{if } \alpha \text{ is acute} \\
      \end{cases}
      \]
      Use this definition to prove a version of the Sine Rule. Suppose that $a$ and $b$ are the lengths of the edges of a triangle opposite angles $\alpha$ and $\beta$ respectively. Then
      \[
      \frac{\sin \alpha}{a} = \frac{\sin \beta}{b}
      \]
      (Hint: Drop a perpendicular from $C$ to $\overline{AB}$ at $D$ and use the fact that you now have two right-triangles. The challenge is to make sure your answer is sufficiently general. Use part (a)…)