Math 161: Homework Questions 5

Submit your answers to questions 1, 2, 4 and 6 at the discussion on Thursday 1st March. All questions except the first are on hyperbolic geometry: you may assume we are within the context of the Poincaré disk model.

1. Let $ABCD$ be the rectangle with vertices $A = (0,0), B = (4,0), C = (4,3), D = (0,3)$.

Suppose that an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps $ABCD$ to a new rectangle $PQRS$ where

$P = f(A) := (2,4)$ and $R = f(C) := (2,9)$

Find all possible isometries $f$, and the remaining points $Q = f(B)$ and $S = f(D)$ of the new rectangle.

2. Prove Proposition 7.3 from the notes: in particular, make sure you get as far as can be claimed without using the parallel postulate!

3. Prove that a circle intersects the unit circle $x^2 + y^2 = 1$ orthogonally if and only if it has equation

$x^2 + y^2 - 2ax - 2by + 1 = 0$ where $a^2 + b^2 > 1$

4. (a) Find the hyperbolic line in the Poincaré disk model on which lie the points $P = (1/4, 0)$ and $Q = (0, 1/2)$.

(b) Use your answer to find the side lengths of the hyperbolic triangle $\triangle OPQ$ where $O = (0,0)$ is the origin.

(This is a long calculation, so take your time...)

5. Recall an example from class. The hyperbolic line $\ell$ joining the points $P, Q = \left(\frac{1}{2}, \pm \sqrt{\frac{5}{12}}\right)$ is an arc of the circle with equation

$x^2 - \frac{10}{3}x + y^2 + 1 = 0$

(a) Use implicit differentiation to calculate $\frac{dy}{dx}$ and hence show that a tangent vector to $\ell$ at $P$ is $\sqrt{15}i + 7j$

(b) Hence show that the angle $\angle OPQ$ in the hyperbolic triangle $\triangle OPQ$ is $\cos^{-1} \frac{5\sqrt{10}}{16} \approx 8.806^\circ$.

6. Prove that two Saccheri quadrilaterals with congruent summits and summit angles must be congruent. Hint: suppose not and show that you can construct a rectangle.

7. Let $PQ\Omega$ be an Omega-triangle. Prove that the sum of the angles $\angle PQ\Omega$ and $\angle QP\Omega$ is less than $180^\circ$.

8. Prove that the summit is always larger than the base in a Saccheri quadrilateral.