Math 161: Homework Questions 5

Submit your answers to questions 2, 4, 5, 8, and 10 at the discussion on Thursday 28th February.

1. (a) For any vectors \( v, w \in \mathbb{R}^2 \), prove that
\[
  v \cdot w = \frac{1}{2} \left( |v + w|^2 - |v|^2 - |w|^2 \right)
\]

(b) Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear map which preserves length. Use (a) to prove that \( f(v) \cdot f(w) = v \cdot w \) for all vectors \( v, w \). Hence conclude that \( f \) preserves angles between vectors.

2. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the isometry, “reflect across the line through the origin making angle 60° with the positive x-axis.”
   (a) Find a \( 2 \times 2 \) matrix \( A \) such that \( f(x) = Ax \).
   (b) Find an alternative written description of \( f \) using only rotations and reflection across the x-axis. Check that your description is correct using matrix multiplication.

3. Recall that every isometry can be viewed as a function (acting on position vectors)
\[
f_{A,b} : x \mapsto Ax + b
\]
where \( A \) is an orthogonal matrix and \( b \) is a constant vector. That is, every isometry is a combination of a rotation/reflection and a translation.

   (a) Prove that isometries obey the composition law
\[
f_{A,b} \circ f_{C,d} = f_{AC,Ad+b}
\]
so that the composition of two isometries is an isometry.

   (b) What is the inverse of the isometry \( f_{A,b} \)? Otherwise said, if \( f_{A,b} \circ f_{C,d} = f_{I,0} \), where \( I \) is the identity matrix, what are \( C \) and \( d \)?

   (c) Compute the composition
\[
f_{A,b} \circ f_{I,d} \circ f_{A,b}^{-1}
\]
(You should obtain a pure translation. For those who’ve done group theory, this shows that the translations form a normal subgroup of the group of isometries.)

4. Let \( ABCD \) be a rectangle with vertices
\[
A = (0,0), \quad B = (4,0), \quad C = (4,3), \quad D = (0,3)
\]
Suppose an isometry \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) maps \( ABCD \) to a new rectangle \( PQRS \) where
\[
P = f(A) := (2,4) \quad \text{and} \quad R = f(C) := (2,9)
\]
Find all possible isometries \( f \), and the remaining points \( Q = f(B) \) and \( S = f(D) \) of the new rectangle.
The remaining questions are within the Poincaré disk model of hyperbolic geometry.

5. Prove Lemma 6.3 from the notes:

(a) If one bisects the base and summit of a Saccheri quadrilateral, one obtains two congruent Lambert quadrilaterals.
(b) The summit angles of a Saccheri quadrilateral are congruent.

In particular, make sure you can do this without using the parallel postulate!

6. Prove the remaining part of Lemma 6.4: a Lambert quadrilateral in Euclidean geometry is a rectangle.

7. Prove carefully that a circle intersects the unit circle \( x^2 + y^2 = 1 \) at right-angles if and only if it has equation

\[
x^2 + y^2 - 2ax - 2by + 1 = 0 \quad \text{where} \quad a^2 + b^2 > 1
\]

8. (a) Find the hyperbolic line in the Poincaré disk model on which lie the points \( P = (1/4, 0) \) and \( Q = (0, 1/2) \).

(b) Find the side lengths of the hyperbolic triangle \( \triangle OPQ \) where \( O = (0, 0) \) is the origin.

(c) The triangle in part (b) is right-angled at \( O \). If \( o, p, q \) represent the hyperbolic lengths of the sides opposite \( O, P, Q \) respectively, check that the Pythagorean theorem \( p^2 + q^2 = o^2 \) is false. Now compute \( \cosh p \cosh q \): what do you observe?

9. In this question we prove Lemma 6.7 from the notes: Fix \( P \) and a hyperbolic line through \( P \). Let \( Q \) lie on this line. Then the distance function \( d(P, Q) \) maps the set of points on one side of \( P \) differentiably and bijectively onto the interval \((0, \infty)\).

The fact that \( d(P, Q) \) is a differentiable function should be clear, since it is a differentiable (multivariable) function of both \( P \) and \( Q \).

Now suppose \( Q \) lies on the hyperbolic line through \( P \) with omega-points \( \Omega, \Theta \). Let \( |Q\Omega| = x \) and \( |Q\Theta| = y \) be Euclidean distances. WLOG assume \( Q \) lies between \( P \) and \( \Theta \) as in the picture.

(a) Show that \( \ln \frac{|P\Theta||Q\Theta|}{|P\Omega||Q\Omega|} > 0 \) so that

\[
d(P, Q) = \ln \frac{|P\Theta|}{|P\Omega|} + \ln x - \ln y
\]

(b) Prove that \( \frac{d}{dQ}d(P, Q) > 0 \) so that \( d(P, Q) \) is a strictly increasing function as \( Q \) moves away from \( P \).

(c) Use the intermediate value theorem to prove that \( d(P, Q) \) is surjective onto \((0, \infty)\).
10. (a) Show that the hyperbolic line $\ell$ joining the points $P, Q = \left( \frac{1}{2}, \pm \sqrt{\frac{5}{12}} \right)$ is an arc of the circle with equation

$$x^2 - \frac{10}{3} x + y^2 + 1 = 0$$

(b) Use implicit differentiation to calculate $\frac{dy}{dx}$ and hence show that a tangent vector to $\ell$ at $P$ is $\sqrt{15}i + 7j$

(c) Hence show that the angle $\angle OPQ$ in the hyperbolic triangle $\triangle OPQ$ is $\cos^{-1} \frac{5\sqrt{10}}{16} \approx 8.806^\circ$, as claimed.