Math 161 Modern Geometry Homework Questions 6

Submit your answers to questions 1(a), 2(a,b), 3(a,b,c) at the discussion on Tuesday 10th March

1. (a) Suppose that the area of a triangle in hyperbolic geometry equals its angular-defect in radians. Let \(ABCD\) be a Lambert quadrilateral with non-right-angle \(\alpha\) at \(D\) (i.e. \(\angle ADC = \alpha < 90^\circ\)). What is its area in terms of \(\alpha\)?
(b) Suppose, in the Poincaré model, that \(B\) is the origin and that \(C = (h, 0)\) and \(A = (0, h)\) for some positive \(h\). What is \(\lim_{h \to 0} \alpha\)?
(c) Suppose that the geometry of the Universe is hyperbolic in nature. With reference to your answer to part (b), explain why any experiments we conduct on Earth might find it hard to justify this.
(d) Compute \(\lim_{\alpha \to 0} h\). Hence draw a Lambert quadrilateral with maximum area.
(e) Argue that a symmetric Lambert quadrilateral \(|BC| = |AB|\) has maximum (hyperbolic) side length \(\ln(\sqrt{2} + 1)\).

2. Recall that the set of orientation-preserving isometries of the Poincaré model are those functions of the form

\[ f(z) = \beta \frac{z - \alpha}{\bar{\alpha}z - 1} \]

where \(|\alpha| < 1\), \(|\beta| = 1\)

Note that \(z = x + iy\), \(\alpha\), \(\beta\) are complex numbers. All other isometries have the form \(f \circ g\) where \(g(z) = \overline{z}\) is the complex conjugate.

(a) Show that there are exactly two orientation-preserving isometries which move the point \(\frac{1}{2}\) to the origin and the point \(\frac{1}{2}i\) to the \(x\)-axis. Where can the point \(\frac{1}{2}i\) end up?
(b) Use your answer to the previous question to compute the hyperbolic distance from \(\frac{1}{2}\) to \(\frac{1}{2}i\).
(c) The hyperbolic Pythagorean Theorem has the form

\[ \cosh c = \cosh a \cosh b \]

where \(a, b, c\) are the hyperbolic side-lengths of a right-angled triangle, with \(c\) the hypotenuse. Use the fact that

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

to check your answer to the previous question.
(d) Look up the power series for hyperbolic cosine and multiply out the first few terms of the Pythagorean Theorem. Show that, as the side-lengths of a right-angled hyperbolic triangle go to zero, the triangle approaches satisfying the Euclidean Pythagorean Theorem \(c^2 = a^2 + b^2\).
3. In class we saw that the hyperbolic area $A$ of a region $R$ in the Poincaré disk is given by

$$A = \int \int_R \frac{4 \, dx \, dy}{(1 - x^2 - y^2)^2} = \int \int_R \frac{4r \, dr \, d\theta}{(1 - r^2)^2}$$

where $(r, \theta)$ are Euclidean polar co-ordinates. We also saw that the arc-length $L$ of a parameterized curve $z(t)$ for $a \leq t \leq b$ is given by

$$L = \int_a^b \frac{2 \left| z'(t) \right|}{1 - \left| z(t) \right|^2} \, dt$$

(a) If $\rho = \ln \left( \frac{1 + r}{1 - r} \right)$ is the hyperbolic distance of a point from the origin, prove that

$$A = \int \int_R \sinh \rho \, d\rho \, d\theta$$

(b) Suppose that $R$ is a circular region centered at the origin with hyperbolic radius $\rho$. Compute its area and circumference.

(c) Use power series expressions for the hyperbolic trig functions to compare your answers to with the area and circumference of a Euclidean circle of radius $\rho$. Show that both the circumference and area of hyperbolic circles are larger than their Euclidean counterparts.

(d) Hyperbolic space is described by geometers as negatively curved. Euclidean space is flat (zero curvature). Spherical geometry is positively curved. Using the definition of radians, a circle drawn on the surface of the unit sphere with center at the north pole and spherical radius $\phi$ consists of those points whose position vectors make an angle $\phi$ with the vertical axis: draw a picture to convince yourself of this! What are the circumference and area of this circle? Are they greater or less than $2\pi \phi$ and $\pi \phi^2$ respectively? Why are you not surprised by your answer?