Math 162A: Differential Geometry Homework 3

Hand in questions 2(a) & 3 at, or before, the lecture on Friday October 17th.

1. Suppose that \( f(t) \) is the function such that \( f'(t) = \sqrt{1 - e^{-2t}} \) and \( f(0) = 0 \), and define the curve

\[
\mathbf{x}(t) = \begin{pmatrix}
\frac{1}{\sqrt{2}} e^{-t} \cos t \\
\frac{1}{\sqrt{2}} e^{-t} \sin t \\
f(t)
\end{pmatrix}.
\]

(\( \mathbf{x}(t) \) is a logarithmic spiral that moves upwards as it spirals towards the center of the \( x, y \)-plane)

(a) Verify that \( \mathbf{x}(t) \) is unit speed.

(b) Calculate the curvature of \( \mathbf{x} \) and show that \( \lim_{t \to \infty} \kappa(t) = 0 \).

2. Consider the following curves:

(a) \( \mathbf{x}(t) = \begin{pmatrix} 4a \cos^3 t \\
4a \sin^3 t \\
3b \cos 2t \end{pmatrix} \),

(b) \( \mathbf{x}(t) = \begin{pmatrix} a(t - \sin t) \\
a(1 - \cos t) \\
4a \sin(t/2) \end{pmatrix} \),

where \( a, b \) are constants. Find the Frenet frame and the curvature and torsion of (a), and just the curvature of (b).

3. Suppose that \( \mathbf{x} : I \to \mathbb{E}^3 \) is a curve which which lies on the surface of the unit sphere (i.e. \( |\mathbf{x}|^2 = 1 \)).

(a) Show that \( \mathbf{x}'' \cdot \mathbf{x} = -1 \) if \( \mathbf{x} \) is unit speed.

(b) Hence or otherwise, prove that the curvature of \( \mathbf{x} \) is at least 1 everywhere.

4. Assume that all the normals of a curve in \( \mathbb{E}^3 \) pass through a fixed point. Show that the curve is contained in a circle.

5. Find all the functions \( f(\theta) \) such that \( \mathbf{x}(\theta) = \begin{pmatrix} a \cos \theta \\
a \sin \theta \\
f(\theta) \end{pmatrix} \) is a plane curve.

6. Suppose that \( \mathbf{x}(t) \) and \( \mathbf{y}(t) = \mathbf{x}(t) + d \mathbf{N}(t) \) are unit speed curves such that the normal vectors \( \mathbf{N} \) of \( \mathbf{x} \) are also the binormal vectors of \( \mathbf{y} \). Prove that the distance \( d \) between the corresponding points of the curves is constant. Prove also that the curvature and torsion of \( \mathbf{x} \) satisfy \( \kappa = \frac{1}{2} d (\kappa^2 + \tau^2) \).