Math 162A: Differential Geometry Homework 4

Hand in questions 2, 3 & 4 at, or before, the lecture on Friday October 24th.
Question 6 is just harder and 7 & 8 are much harder than A grade work.
No homework week 5 because of the midterm on Monday — use this to prepare!

1. Find the curvature of the degree \( n \geq 1 \) polynomial \( y = x^n \) at all points (find a sensible parameterization of the curve first). For each \( n \) find the maximum value of the curvature and the location on the curve of the point of maximum curvature (it’s messy...).

2. Recall the tractrix
\[
x(t) = \left( \frac{\sinh^{-1} t - t(1 + t^2)^{-1/2}}{(1 + t^2)^{-1/2}} \right).
\]
Show that the tangent line to the tractrix hits the \( x \)-axis at a distance 1 from the curve for \( t > 0 \) (this justifies the claim that the tractrix is the curve traced by a mass starting at \( \left( \frac{0}{1} \right) \) attached by a length 1 rope to a vehicle moving along the \( x \)-axis).

3. Find the curvature of the tractrix in the above parameterization for \( t > 0 \).

4. Show directly that the tangent vector at time \( t \) of the involute \( i(t) = x(t) - tx'(t) \) of a unit speed curve is always orthogonal to the tangent vector of the original curve at time \( t \).

5. Show that the evolute of the ellipse
\[
x(t) = \left( \frac{a \cos t}{b \sin t} \right)
\]
is given by \( e(t) = (a^2 - b^2) \left( \frac{a^{-1} \cos^3 t}{-b^{-1} \sin^3 t} \right) \).
This curve is a called an asteroid (have a look at the animation on the Wikipedia evolute page!).

6. Suppose that \( x(t) \) is any unit speed curve in \( \mathbb{E}^2 \) with positive curvature everywhere. Prove that the curvature of its involute is \( \frac{1}{t} \). It looks like all involutes have the same curvature, and so all involutes should be identical up to rigid motions. Explain why this conclusion is nonsense.

7. A cylinder is any surface \( S(t,s) = y(t) + sk \), where \( k \) is a constant vector, and \( y(t) \) is a curve in the space \( k^\perp \). A cylindrical (but not necessarily circular) helix is any curve \( x(t) \) such that the unit tangent vector \( T \) makes a constant angle with some fixed vector \( k \). Suppose that \( x \) is a unit speed cylindrical helix, and that the angle in question is not a right angle.

   (a) Show that \( x \) is a curve lying on the surface of a cylinder.

   (b) Use the Frenet-Serret formulas to prove that curve is a cylindrical helix iff \( \kappa / \tau \) is constant.

8. Suppose that a moving frame \( e_1, e_2, e_3 \) has structure equations such that all three functions \( \bar{w}_{12}, \bar{w}_{13}, \bar{w}_{23} \) are constant. Find the moving frame \( f_1, f_2, f_3 \) where \( f_1 = e_1 \) such that \( f_1, f_2, f_3 \) is the Frenet frame of a circular helix. Calculate the curvature \( \kappa \) and torsion \( \tau \) of this helix in terms of \( \bar{w}_{12}, \bar{w}_{13}, \bar{w}_{23} \). Can you write down an orthogonal matrix \( A \) such that
\[
A^{-1} \begin{pmatrix} 0 & \bar{w}_{12} & \bar{w}_{13} \\ -\bar{w}_{12} & 0 & \bar{w}_{23} \\ -\bar{w}_{13} & -\bar{w}_{23} & 0 \end{pmatrix} A = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}?
\]