Math 162A: Homework 6

Submit your answers to the starred questions at the discussion class on Thursday 8th March.

1. Compute the first fundamental forms of the following parameterized surfaces wherever they are regular ($a, b, c$ are constants):
   
   (a) Ellipsoid $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)^T$
   
   (b) Elliptic paraboloid $\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)^T$
   
   (c) Hyperbolic paraboloid $\mathbf{x}(u, v) = (au \cosh v, bu \sinh v, u^2)^T$
   
   (d) Hyperboloid of two sheets $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)^T$

   How do these simplify in the symmetric cases when $a = b = c$? Where does each parameterization fail to be regular?

2. * Calculate the fundamental forms of Enneper’s surface
   
   $\mathbf{x}(u, v) = \begin{pmatrix} u - u^3/3 + uv^2 \\ v - v^3/3 + vu^2 \\ u^2 - v^2 \end{pmatrix}$

3. * Consider the upper half plane $y > 0$ equipped with the first fundamental form $I = \frac{dx^2 + dy^2}{y^2}$ as in the notes. Compute the arc-length between the points $(1, 1)$ and $(-1, 1)$ in two ways:
   
   (a) Over the circular arc centered at the origin.
   
   (b) Over the straight line between the points.

   Compare your answers!

   Also compute the arc-length over the circular arc centered at the origin between the points $(r, r)$ and $(-r, r)$. How does this arc-length depend on $r$? (Quote the integral from the notes if you wish)

4. * Let $y(s)$ be a unit speed biregular curve and $\mathbf{x}(s, t) = y(s) + ty'(s)$ be its tangent developable, where $t > 0$. Calculate the first and second fundamental forms of $\mathbf{x}$ in terms of the parameterization $s, t$ and the curvature and torsion of $y$.

   Hence or otherwise show that the fundamental forms of the tangent developable of the unit helix $y(s) = \begin{pmatrix} \cos s/\sqrt{2} \\ \sin s/\sqrt{2} \\ s/\sqrt{2} \end{pmatrix}$ are

   $I = \left(1 + \frac{t^2}{4}\right) ds^2 + 2 ds dt + dt^2, \quad \mathcal{I} = -\frac{t}{4} ds^2$

5. (Hard) Consider the parameterization of the torus with interior hole of radius 1 and rotating circle also of radius 1.

   $\mathbf{x}(u, v) = \begin{pmatrix} (2 + \cos u) \cos v \\ (2 + \cos u) \sin v \\ \sin u \end{pmatrix}, \quad u, v \in \mathbb{R}$

   Consider also the curve $z(t) = (t, \alpha t)$ in $\mathbb{R}^2$, where $\alpha \in \mathbb{R}$ is constant, so that $\mathbf{x}(z(t))$ is a curve on the torus.
(a) Prove that $x(z(t))$ has a self-intersection ($\exists s \neq t$ such that $x(z(t)) = x(z(s))$) if and only if $\alpha$ is a rational number.

(b) If $\alpha \in \mathbb{Q}$, prove that the curve is closed. Moreover show that there exists a minimum positive value of $t$ for which $x(z(t)) = x(z(0))$ and find it in terms of $\alpha$.

(Recall that a closed curve has no ends, it keeps tracing over itself indefinitely — e.g. a circle)

(c) Thus write down (but don’t evaluate!) the integral for the length of the closed curve in terms of $\alpha$. 