Math 162A: Differential Geometry Midterm

Total marks = 50. When a question asks you to sketch a curve, you are not required to calculate exactly, just indicate roughly what the curve looks like.

1. State what it means for a curve \( x : \mathbb{R} \rightarrow \mathbb{R}^3 \) to be regular. What is required in addition for \( x \) to be biregular? One each of the following three curves is regular, biregular and neither at \( t = 0 \): without calculating anything, state which is which. (10)

\[
\begin{align*}
\text{(a)} \quad & x(t) = \begin{pmatrix} t \\ |t| \\ 1 \end{pmatrix}, \\
\text{(b)} \quad & y(t) = \begin{pmatrix} t \\ t|t| \\ 1 - t \end{pmatrix}, \\
\text{(c)} \quad & z(t) = \begin{pmatrix} t \\ \cos|t| \\ t^2 \end{pmatrix}.
\end{align*}
\]
2. Consider the curve \( \mathbf{x}(t) = \begin{pmatrix} 3t \\ t^3 \\ 3t \end{pmatrix} \).

(a) Find the unit tangent vector and the curvature of \( \mathbf{x}(t) \). (10)

(b) Without calculating either the unit normal or binormal vectors, argue that the torsion of \( \mathbf{x}(t) \) is zero. (4)

(c) Argue, given your answers to parts (a) and (b), that you could write down two possibilities for the Frenet frame of \( \mathbf{x}(t) \) \textit{without} performing any further differentiations. (6)
3. Recall that the evolute \( e(t) \) and involute \( i(t) \) of a planar curve \( x(t) \) are defined by

\[
e(t) = x(t) + \frac{1}{\kappa(t)} N(t), \quad i(t) = x(t) - tx'(t),
\]

where \( x \) must be unit speed for the involute to be correctly defined.

(a) On the following plot of a letter ‘e’, draw your best guesses for all the osculating circles at the intersection point \( A \), the point (seemingly) of maximum curvature \( B \), and the end of the tail \( C \). Using these sketch your best guess for the evolute of ‘e’. Also, supposing that \( t = 0 \) is at point \( B \), and that \( t \) increases as you move down and right from \( B \), draw your guess at the involute for this parameterization and for \( t > 0 \).
(b) We have seen that the evolute $e(t)$ of a curve $x(t)$ is the focal line of its normal vector field. This, together with looking at pictures, suggests that, at places where $e$ is differentiable, there should be a function $\alpha(t)$ such that

$$e(t) + \alpha(t)e'(t) = x(t).$$

Suppose that $x(t)$ is a curve in $\mathbb{R}^2$ with non-constant curvature. Find $\alpha(t)$. \hfill (13)