Math 162B: Differential Geometry Homework 1

Hand in questions 4, 5 and 6 at lecture Monday 12th January.

First a couple of questions to remind yourself about 162A...

1. Consider the curve $\mathbf{x}(t)$ defined by

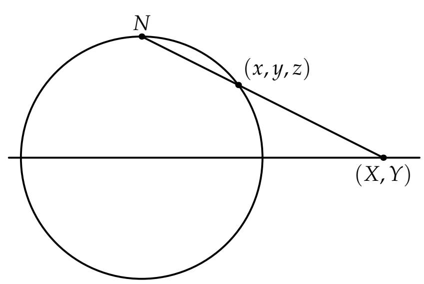
$$\mathbf{x}(t) = \begin{cases} \begin{pmatrix} t \\ t^3 \\ 0 \end{pmatrix} & \text{when } t > 0, \qquad \begin{pmatrix} t \\ 0 \\ t^3 \end{pmatrix} \text{ when } t < 0, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ when } t = 0. \end{cases}$$

   (a) Prove that $\mathbf{x}(t)$ is a regular differentable curve.
   (b) Show that the torsion of $\mathbf{x}(t)$ is zero whenever it is defined.
   (c) $\mathbf{x}(t)$ is not a planar curve. Given part (b), explain how this is possible.

2. Recall from the lectures the comment that circles on the unit sphere $x^2 + y^2 + z^2 = 1$ are mapped under the stereographic projection

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{1-z} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad z \neq 1,$$

to circles or lines in the equatorial plane. Prove this theorem using the following steps.



Recall that a circle on the surface of the sphere is the intersection of the sphere with a plane.

   (a) Assume that $(x, y, z)^T$ lies on the plane $ax + by + cz = c$. Prove that $aX + bY = c$.
   (b) Prove that $X^2 + Y^2 = \frac{1+z}{1-z}$.
   (c) Suppose now that $(x, y, z)^T$ lies on the plane $ax + by + cz = 1 + c$. Use part (b) to show that in this case (X, Y) satisfies the equation of a circle

$$(X - a)^2 + (Y - b)^2 = a^2 + b^2 - 2c - 1. \qquad (\dagger)$$

It remains to show two things: that we have taken into consideration all circles on the sphere, and that the circle defined in part (c) really is a circle.

   (d) Consider the plane $ax + by + cz = d$. If $c \neq d$, show that the equation of the plane may be rewritten so that $d - c = 1$. Argue that all planes in $\mathbb{R}^3$ are considered in parts (a) and (c).

(e) Assume that the plane $ax + by + cz = d$ intersects the unit sphere. By considering the unit normal vector $\frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)^T$ to the plane, explain why we must have

$$|d| \leq \sqrt{a^2 + b^2 + c^2}.$$

What does equality in this formula mean? Use this to conclude that the right hand side of (†) is non-negative.

(f) Use parts (a) through (e) to argue that any circle on the surface of the sphere is projected to a circle or a line under the stereographic projection and that conversely every circle and line in the equatorial plane arises this way.

3. Consider the following differential forms in $\mathbb{R}^3$.

$$\alpha = 2xy\,\mathrm{d}x + 3xz\,\mathrm{d}y - \mathrm{d}z, \qquad \beta = z\,\mathrm{d}x - z^3\,\mathrm{d}y - 2\,\mathrm{d}z,$$
$$\gamma = y\,\mathrm{d}x \wedge \mathrm{d}y - xy^2\,\mathrm{d}x \wedge \mathrm{d}z - \mathrm{d}y \wedge \mathrm{d}z,$$
$$\delta = -yz^2\,\mathrm{d}x \wedge \mathrm{d}y - 3xy\,\mathrm{d}x \wedge \mathrm{d}z + z^2\,\mathrm{d}y \wedge \mathrm{d}z.$$

Find every possible wedge product of two of the forms, in both orders, writing the results in standard form. Check, for each pair ω, ϕ of forms, that $\omega \wedge \phi = (-1)^{\deg\omega \deg\phi}\phi \wedge \omega$ is satisfied.

4. Calculate the exterior derivative of each of the forms in Exercise 3, giving your answers in standard form. Compute $\mathrm{d}(\alpha \wedge \beta)$ directly, and check that it equals $\mathrm{d}\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \mathrm{d}\beta$. Check also that $\mathrm{d}(\mathrm{d}\alpha) = 0$.

5. Let r, θ, ϕ be spherical polar co-ordinates: i.e. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Compute $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ in terms of $\mathrm{d}r$, $\mathrm{d}\theta$, $\mathrm{d}\phi$. Moreover, show that

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

6. Let f, g be functions and consider the 1-form $\alpha = g\,\mathrm{d}f$. Show that $\alpha \wedge \mathrm{d}\alpha = 0$. Is it possible to write $\mathrm{d}x + y\,\mathrm{d}z$ in $\mathbb{R}^3$ in the form $g\,\mathrm{d}f$?

7. Let α, β be 1-forms in $\mathbb{R}^n$, with α everywhere non-zero. Show that if $\alpha \wedge \beta = 0$ then β is proportional to α (i.e. there is a function f such that $\beta = f\alpha$). (Hint: $\alpha \neq 0$ everywhere, so extend $\alpha|_p$ to a basis of 1-forms at p. Now write $\beta|_p$ in terms of this basis…)