Math 162B: Differential Geometry Homework 2

Hand in questions 2 and 5 at lecture Wednesday 21st January.

1. Consider the vector fields
$$u = x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \qquad v = y\frac{\partial}{\partial x} - xy\frac{\partial}{\partial z},$$
and the 2-forms
$$\alpha = y\,\mathrm{d}x\wedge\mathrm{d}y - z\,\mathrm{d}x\wedge\mathrm{d}z, \qquad \beta = 3\,\mathrm{d}x\wedge\mathrm{d}z - yz\,\mathrm{d}y\wedge\mathrm{d}z,$$
$$\gamma = z\,\mathrm{d}x\wedge\mathrm{d}y - y\,\mathrm{d}x\wedge\mathrm{d}z + z\,\mathrm{d}y\wedge\mathrm{d}z.$$
Calculate $\alpha(u,v)$, $\beta(u,v)$, $\gamma(u,v)$.

2. Let $\mathbf{x}(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$.
   (a) Show that
   $$\mathbf{e}_1 := \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \mathbf{e}_2 := \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad \mathbf{e}_3 := \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$
   is a moving frame.
   (b) Calculate $\mathrm{d}\mathbf{x}$ in terms of the moving frame above, in particular finding the 1-forms $\theta_1, \theta_2, \theta_3$. (*Be careful not to confuse the co-ordinate/angle θ with the 1-forms θ_i*).
   (c) Compute the connection 1-forms $\omega_{12}, \omega_{13}, \omega_{23}$.
   (d) Check explicitly that the first and second structure equations hold.
   (e) Suppose that you knew nothing about $\mathbf{x}$, or the moving frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and that you only have your computed 1-forms $\theta_1, \theta_2, \theta_3$. Solve for the connection 1-forms ω_{ij} just by using the first structure equations.

3. Continuing from the previous question, suppose that we identify 1-forms and 2-forms in the co-ordinates r, θ, ϕ with traditional vector fields as follows:
$$\alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\theta_3 \longleftrightarrow \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3,$$
$$\beta_1\theta_2\wedge\theta_3 + \beta_2\theta_3\wedge\theta_1 + \beta_3\theta_1\wedge\theta_2 \longleftrightarrow \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2 + \beta_3\mathbf{e}_3,$$
and 3-forms with functions; $\gamma\theta_1\wedge\theta_2\wedge\theta_3 \longleftrightarrow \gamma$. Use the expressions you found for $\theta_1, \theta_2, \theta_3$ to show that, in spherical polar co-ordinates, we have
$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_1 + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_2 + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_3,$$
$$\nabla\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial v_1 r^2}{\partial r} + r\frac{\partial v_2\sin\theta}{\partial\theta} + r\frac{\partial v_3}{\partial\phi}\right).$$
Feel free to calculate the curl too, though it's always much messier!

4. Suppose that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a moving frame of $\mathbb{E}^3$, and that $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a map into $\mathbb{E}^3$. Prove that

$$\theta_1 \wedge \theta_2 \wedge \theta_3 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

regardless of the choice of moving frame. (*You will need to compute θ_i in terms of* $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ *and the entries of* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \ldots$)

5. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{E}^2$ be a smooth map with co-ordinate functions u, v, and suppose that $\mathbf{x}_u, \mathbf{x}_v$ are everywhere orthogonal, so that $\mathbf{x}_u = h\mathbf{e}_1$ and $\mathbf{x}_v = k\mathbf{e}_2$ for functions h, k and a moving frame $\mathbf{e}_1, \mathbf{e}_2$ (h, k are the norms of $\mathbf{x}_u, \mathbf{x}_v$ up to sign). Use the structure equations (calculate $\theta_1, \theta_2, \omega_{12}$, etc.) to show that

$$\frac{\partial}{\partial v}\left(\frac{h_v}{k}\right) + \frac{\partial}{\partial u}\left(\frac{k_u}{h}\right) = 0.$$