Math 162B: Differential Geometry Homework 3

Hand in questions 2, 5 & 6 at lecture Monday 2nd February

1. Check that the surfaces

$$\mathbf{x}(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ \ln u \end{pmatrix}, \qquad \mathbf{y}(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ v \end{pmatrix},$$

have the same Gauss curvature at all points of $U = \{(u,v) \in \mathbb{R}^2 : v > 0\}$, but that $\mathbf{x}, \mathbf{y}$ are not isometric. *This example shows that Gauss' Theorem Egregium has no converse.*

2. Let $\mathbf{x}(s,t) = \mathbf{y}(s) + t\mathbf{y}'(s)$ be the tangent developable of a unit speed curve $\mathbf{y}(s)$ in $\mathbb{E}^3$. Show that $\mathbf{e}_1 = \mathbf{y}'(s)$, $\mathbf{e}_2 = \kappa^{-1}\mathbf{y}''(s)$ are orthogonal vector fields tangent to the surface. Calculate $\theta_1, \theta_2, \omega_{12}$ for these vector fields, and use these to prove that all tangent developables are flat.

3. Consider the metric $\mathrm{I} = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$ in polar co-ordinates on $\mathbb{R}^2$ with the origin removed. Show that $K = 0$. Why is this to be expected?

4. Let $f(x)$ and $g(y)$ be smooth functions. Find the Gauss curvatures of the following metrics:

   (a) $\mathrm{I}_1 = f(x)^2\,\mathrm{d}x^2 + g(y)^2\,\mathrm{d}y^2$,

   (b) $\mathrm{I}_2 = g(y)^2\,\mathrm{d}x^2 + f(x)^2\,\mathrm{d}y^2$.

5. x, y are Tchebyshev co-ordinates for a surface $\mathbf{x} : U \to \mathbb{E}^3$ if $\mathrm{I} = \mathrm{d}x^2 + 2\cos\phi\,\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y^2$, where $\phi : U \to \mathbb{R}$ is some function. Show that $\mathbf{x}$ is pseudospherical with $K = -1$ if and only if ϕ satisfies the *Sine–Gordon equation* $\phi_{xy} = \sin\phi$. (*There is no frame given here, so you'll have to attmpt to find suitable θ_1, θ_2 directly: use the lemma from the course perhaps...*)

6. Consider a 2-dimensional simplicifaction of the Schwartzschild metric where the units have been removed: $\mathrm{I} = (1 - r^{-1})\,\mathrm{d}t^2 - (1 - r^{-1})^{-1}\,\mathrm{d}r^2$. Writing $\theta_1 = (1 - r^{-1})^{1/2}\,\mathrm{d}t$ and $\theta_2 = (1 - r^{-1})^{-1/2}\,\mathrm{d}r$ we have $\mathrm{I} = \theta_1^2 - \theta_2^2$. The first structure equations in this situation are

$$\mathrm{d}\theta_1 + \omega_{12} \wedge \theta_2 = 0 = \mathrm{d}\theta_2 + \omega_{12} \wedge \theta_1.$$

   There is no sign error! The equations are different because the metric is not positive definite: indeed $\omega_{12} = \omega_{21}$! Use the equation $\mathrm{d}\omega_{12} = K\theta_1 \wedge \theta_2$ to show that the curvature is well-defined at $r = 1$ (the Schwartzschild radius of this example), only being undefined at $r = 0$ (the center of the black hole).

7. The *Poincaré disc* is an alternative model of hyperbolic space: equip the inside of the unit circle $r < 1$ with $\mathrm{I} = \frac{4}{(1-r^2)^2}(\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$.

   (a) Show that this space has $K = -1$.

   (b) (Only for those who like complex analysis) You are given that the map $x + iy = z \mapsto \frac{z-i}{z+i} = re^{i\phi}$ is a bijective mapping of the upper half plane to the inside of the unit disc (it's a Möbius transform). Prove that this map is an isometry between the hyperbolic space given in lectures and the Poincaré disc.

8. (HARD) Suppose that a domain $U \subset \mathbb{R}^2$ is equipped with a metric I. Let $\mu : U \to \mathbb{R}$ be a smooth function and define a new *conformal* metric $\hat{\mathrm{I}} := e^{2\mu}\mathrm{I}$ on U. Prove that

$$\hat{K} = e^{-2\mu}(K - \triangle\mu),$$

where $\triangle\mu = v_1[v_1[\mu]] + v_2[v_2[\mu]]$ is the Laplacian (v_1, v_2 are any two vector fields which are orthonormal for the fundamental form I). (*Given any K the equation $\triangle\mu = K$ has a solution u: this says that all metrics in 2-dimensions are conformally flat*.)