## Math 162B: Differential Geometry Homework 4

Hand in questions 2, 3 \& 4 at lecture Wednesday 18th February

1. Suppose that a surface has first fundamental form $\mathrm{I}=\mathrm{d} u^{2}+\mathrm{d} v^{2}$ and second fundamental form $\mathbb{I I}=k_{1} \mathrm{~d} u^{2}+k_{2} \mathrm{~d} v^{2}$. Prove that $k_{1}$ is a function only of $u$, that $k_{2}$ is a function only of $v$, and that at least one of $k_{1}, k_{2}$ is always zero.
2. For the standard surface of revolution as parameterized in the lectures, show that the conserved quantity $A=f^{2} \phi^{\prime}$ is in fact equal to $f \cos \alpha$, where $\alpha$ is the angle between the geodesic and the lines of latitude ( $u=$ constant). This fact is called Clairut's theorem.
3. Use the previous exercise to quickly give a description of the geodesics on a cylinder.
4. Let $\mathbf{x}$ be a surface of revolution with $f(u)=\sqrt{2+2 u^{2}}$ and let $\mathbf{x}(z(t))$ be the geodesic starting at $\left(\begin{array}{l}2 \\ 0 \\ 1\end{array}\right)$ with initial velocity $\left(\begin{array}{c}1 / 2 \\ 1 / \sqrt{2} \\ 1 / 2\end{array}\right)$. Prove that the geodesic makes angle $\alpha=\tan ^{-1} u$ with the lines of latitude.
5. Show that a general geodesic (not a straight line) on a cone $z^{2}=a^{-2}\left(x^{2}+y^{2}\right)$ will have $n$ selfintersections iff $\frac{1}{4(n+1)^{2}-1} \leq a^{2}<\frac{1}{4 n^{2}-1}$. (Normalize the formulae in the notes by setting $B=C=$ 0 and rescaling $t$, then find all the solutions $t_{1}, t_{2}$ to $\left(u\left(t_{1}\right), \phi\left(t_{1}\right)\right)=\left(u\left(t_{2}\right), \phi\left(t_{2}\right)\right)$. What are the restrictions on $\phi(t)$ ?)
6. (a) Consider a local surface such that $\mathbf{x}_{u} \cdot \mathbf{x}_{v}=0$. Choose the adapted frame such that

$$
\mathbf{x}_{u}=p \mathbf{e}_{1}, \quad \mathbf{x}_{v}=q \mathbf{e}_{2},
$$

where $p, q$ are positive functions on $U$. Calculate $\theta_{1}, \theta_{2}$ for this parameterization, and hence find an expression for the Gauss curvature $K$ in terms of $p, q$ and their $u, v$-derivatives.
(b) Think of the $v=$ constant curves as being a family of curves parameterized by $u$ and suppose that we have chosen our parameters $u, v$ such that the curves $v=$ constant are geodesics: i.e. the curves $\mathbf{y}(t)=\mathbf{x}(u(t), v)$ for $v$ constant are geodesics. The vector $\mathbf{x}_{v}$ then 'connects' you orthogonally from one geodesic to an infinitessimally nearby one. Write down the geodesic and energy equations and show that $p_{v}=0$. Deduce that $q^{\prime}=q_{u} u^{\prime}=$ $q_{u} / p$ along a curve $\mathbf{y}(t)$. Differentiate again to see that

$$
q^{\prime \prime}+K q=0 .
$$

Since $q>0$ we have that $q^{\prime \prime}$ is opposite in sign to $K$. Since $q$ is the infinitessimal distance between neighboring geodesics, we see that regions of positive curvature tend to cause geodesics to come together, while regions of negative curvature tend to separate them.
7. (Hard - only for those who like complex analysis) Recall the transformation between the hyperbolic upper half-plane and the Poincaré disc from homework 3. Use the fact that Möbius transforms preserve circles and angles of intersection to argue that the geodesics for the Poincaré disc are circles which intersect the edge of the disc orthogonally.
8. (Only for the Physicists) In relativity, inertial motion is along geodesics. Int this question, we find the equation of motion for a free-fall geodesic in the Schwarzschild metric and show that we recover Newton's inverse square law of gravitation in the limit. Recall the Schwarzschild radius $R=\frac{2 G m}{c^{2}}$ of a body of mass $m$, where $c$ is the speed of light, and $G$ the gravitational constant.
(a) Consider the (1,1)-Schwarzschild metric $\mathrm{I}=\left(1-R r^{-1}\right)^{-1} \mathrm{~d} r^{2}-c^{2}\left(1-R r^{-1}\right) \mathrm{d} t^{2}$. Write $\mathrm{I}=\theta_{1}^{2}-\theta_{2}^{2}$ and show that $\omega_{12}=\omega_{21}=\frac{c R}{2 r^{2}} \mathrm{~d} t$ (the structure equations here are $\mathrm{d} \theta_{1}+\omega_{12} \wedge$ $\theta_{2}=0=\mathrm{d} \theta_{2}+\omega_{21} \wedge \theta_{1}$ where $\omega_{12}=\omega_{21}$.
(b) Let $z(s)=(r(s), t(s))$ be a geodesic. Primes ' will denote derivatives with respect to $s$. Show that the first geodesic equation $\frac{\mathrm{d}}{\mathrm{ds}} \theta_{1}\left(z^{\prime}\right)+\omega_{12}\left(z^{\prime}\right) \theta_{2}\left(z^{\prime}\right)=0$ is equivalent to

$$
r^{\prime \prime}=\frac{R}{2 r^{2}}\left(\left(1-R r^{-1}\right)^{-1} r^{\prime 2}-c^{2}\left(1-R r^{-1}\right) t^{\prime 2}\right)
$$

(c) Show that the second geodesic equation $\frac{\mathrm{d}}{\mathrm{ds}} \theta_{2}\left(z^{\prime}\right)+\omega_{21}\left(z^{\prime}\right) \theta_{1}\left(z^{\prime}\right)=0$ is equivalent to

$$
\left(1-R r^{-1}\right) t^{\prime}=k \text {, a constant. }
$$

(d) The Energy equation in this context reads $\theta_{1}\left(z^{\prime}\right)^{2}-\theta_{2}\left(z^{\prime}\right)^{2}=-c^{2} \tau^{\prime 2}$ is constant. $\tau$ is the proper time and is the time as measured on a clock falling under the action of the metric. Put the three geodesic equations together to see that

$$
r^{\prime \prime}=-\frac{R c^{2}}{r^{2}} \tau^{\prime 2}
$$

(e) Finally, deduce that, with respect to proper time, a body falling radially in the Schwarzschild metric experiences acceleration according to Newton's inverse square law:

$$
\frac{\mathrm{d}^{2} r}{\mathrm{~d} \tau^{2}}=-\frac{G m}{r^{2}}
$$

