Math 162B: Differential Geometry Homework 5

Hand in questions 1,5 & 6 at lecture Friday 27th February. Questions 8 and 9 are beyond the examinable limits of the course.

1. Consider the torus of revolution, parameterized by

$$\mathbf{x}(u,\phi) = \begin{pmatrix} (a \pm \sqrt{r^2-u^2})\cos\phi \\ (a \pm \sqrt{r^2-u^2})\sin\phi \\ u \end{pmatrix},$$

   where $a > r > 0$ are constants. Appeal to the discussion of the geodesic equations for a surface of revolution to argue that:

   (a) Only two lines of constant height u on the torus are geodesics.

   (b) If a geodesic starts off at a point with $u = r$ tangent to the ϕ co-ordinate curve, then the geodesic will be forever confined to the outer part $f > a$ of the torus.

2. Let $\mathbf{v}(t)$ be the parallel transport of a vector $\mathbf{v}_0$ along a unit speed geodesic $\gamma(t)$. Use the geodesic equations to prove that the angle between $\mathbf{v}(t)$ and $\gamma'(t)$ is constant. (*It is quickest to appeal to question 6 for this, but you can do it in terms of a moving frame*)

3. Consider the unit sphere and the tangent vector $\mathbf{v} = (-1,0,0)^T$ at the north pole $(0,0,1)^T$. Perform the parallel transport around the geodesic triangle as described in the notes (down to the equator, round the equator by ϕ_0 and back to the north pole), and show that the result is the tangent vector $(-\cos\phi_0, -\sin\phi_0, 0)^T$. (*Hint: because of question 2, calculating the parallel transports along these curves is easy and requires no solving of differential equations*)

4. Consider the surface of revolution given by $f(u) = 1+u^2$ and the curve $z(t) = (u(t),\phi(t)) = (t,-t)$. Calculate $\omega_{12}(z')$ for this curve and find $g(t) = \int_0^t \omega_{12}(z')\,\mathrm{d}t$. Hence show that the parallel transport of the tangent vector $\mathbf{v}_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ at $\mathbf{x}(z(0)) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is horizontal at $t = \frac{1}{2}\sqrt{(\pi/2+1)^2 - 1}$.

5. Let $\mathbf{y} : I \to \mathbb{E}^3$ be a biregular curve parameterized by arc-length s. Consider the parameterized surface

$$\mathbf{x}(s,v) = \mathbf{y}(s) + v\mathbf{B}(s), \quad s \in I,\ v \in (-\epsilon,\epsilon),\ \epsilon > 0,$$

   where $\mathbf{B}$ is the binormal vector field of $\mathbf{y}$. For small ϵ, prove that the image of $\mathbf{x}$ is a regular surface S. Morover, prove that $\mathbf{y}$ is a geodesic in this surface (thus all biregular curves are a geodesic in some surface).

6. Let $\mathbf{v},\mathbf{w}$ be vector fields along a curve $\gamma : I \to S$ and $D_{\frac{\mathrm{d}}{\mathrm{d}t}}$ the covariant derivative operator. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v},\mathbf{w}) = \left(D_{\frac{\mathrm{d}}{\mathrm{d}t}}\mathbf{v},\mathbf{w}\right) + \left(\mathbf{v}, D_{\frac{\mathrm{d}}{\mathrm{d}t}}\mathbf{w}\right).$$

7. In this question we show that the only curves of constant geodesic curvature on the sphere are circles.

   (a) Let $\gamma(t)$ be a unit speed curve on the surface of the unit sphere. Argue that the normal curvature of γ is always $\kappa_n = 1$ (look up the definition in the 162A notes if you've forgotten it).
   (b) Show that γ has constant curvature κ if and only if it has constant geodesic curvature κ_g and write down the relation between them.
   (c) Let γ trace out a circle on the surface of the unit sphere. By choosing spherical polar co-ordinates such that the (geodesic) center of the circle γ is the north pole (i.e. γ is the curve $\theta = \theta_0$, where θ_0 is the geodesic radius of γ), find the geodesic curvature of γ in terms of the geodesic radius of the circle.
   (d) Suppose now that γ is a unit speed curve on the surface of the sphere with constant geodesic curvature κ_g. Prove that γ''' is perpendicular to γ and to γ''.
   (e) Argue that γ''' is orthogonal to $D\gamma'$ and thus that it is parallel to γ'.
   (f) Conclude that $\mathbf{k} = \gamma' \times \gamma''$ is constant and thus that γ is a circle on the sphere.

8. Suppose that X, Y are vector fields on U. Show that the Lie bracket $[X, Y] := X \circ Y - Y \circ X$ is a vector field on U (Hint: write $X = x_1 \frac{\partial}{\partial u_1} + x_2 \frac{\partial}{\partial u_2}$ and Y with respect to some co-ordinates u_1, u_2 and calculate...).

9. Extending question 6, prove that the Levi–Civita connection ∇ on U induced by a surface $\mathbf{x}$ (recall $\mathrm{d}\mathbf{x}(\nabla_X Y) = \pi^T \mathrm{d}_X(\mathrm{d}\mathbf{x}(Y))$) is:

   (a) Metric: $\mathrm{d}_X(\mathrm{I}(Y, Z)) = \mathrm{I}(\nabla_X Y, Z) + \mathrm{I}(Y, \nabla_X Z)$;
   (b) Torsion-free: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$.

   (*Think what (a) means in terms of* $\mathrm{I} = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}$, *while taking* $\mathrm{d}\mathbf{x}$ *of (b) will help*)