Math 162B: Differential Geometry Homework 5 $\frac{1}{2}$

A few extra questions, none for submission.

1. (a) Consider the cylinder $\mathbf{x}(u,\phi) = \begin{pmatrix} \cos\phi \\ \sin\phi \\ u \end{pmatrix}$ and the curve $\gamma(t) = \mathbf{x}(z(t))$ where $z(t) = (u(t), \phi(t)) = (t^2, t)$. Compute the covariant derivative of the tangent vector field to the curve γ on the surface of the cylinder.

   (b) Use this to compute the geodesic curvature of γ (use whichever orientation of the cylinder you prefer).

   (c) Find a plane curve $\mathbf{y}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$ (in $\mathbb{E}^2$) whose curvature is the same as the geodesic curvature of γ?

2. (a) Similarly to question 1, show that the geodesic curvature of the curve $\gamma(t) = \mathbf{x}(z(t))$ where $z(t) = (\cos t, \sin t)$ on the surface of the cylinder is 1.

   (b) Given any curve $z(t) = (u(t), \phi(t))$, show by calculation that the plane curvature of $z(t)$ is equal to the geodesic curvature of $\gamma(t) = \mathbf{x}(z(t))$ up to a sign determined by the orientation of the cylinder.

   (c) Give a fast geometric argument for why part (b) is true and why it will not be true for curves on a general surface.

3. Directly calculate the integral of the 1-form $\alpha = xy\,\mathrm{d}y$ over the unit circle in $\mathbb{R}^2$. Why is the answer obvious using Stokes' theorem?

4. Suppose Σ is a regular surface with boundary $\partial\Sigma$ and that Σ has no self-intersections. Suppose also that $\beta = 2xy\,\mathrm{d}x \wedge \mathrm{d}y + 2xz\,\mathrm{d}x \wedge \mathrm{d}z$ is a 2-form on $\mathbb{R}^3$. Show that the integral of β over Σ depends only on the boundary $\partial\Sigma$. Calculate $\int_\Sigma \beta$ when $\partial\Sigma$ is the circle $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, x - z = 0\}$.

5. (Harder - do question 7 on sheet 6 first) Use the integral of the 1-form $\alpha = x\,\mathrm{d}y$ to calculate the area of a regular n-gon whose vertices lie on a circle of radius 1. Compute the side length of a regular 17-gon of area 1. *Hint: Let the corners of the n-gon have co-ordinates $c_k = (\cos\frac{2\pi}{n}k, \sin\frac{2\pi}{n}k)$ for $k = 0, \ldots, n-1$. First calculate α restricted to the kth edge joining c_k, c_{k+1}.*