Math 162B: Differential Geometry Homework 5 1/2

A few extra questions, none for submission.

1. (a) Consider the cylinder $\mathbf{x}(u, \phi) = \begin{pmatrix} \cos \phi \\ \sin \phi \\ u \end{pmatrix}$ and the curve $\gamma(t) = \mathbf{x}(z(t))$ where $z(t) = (u(t), \phi(t)) = (t^2, t)$. Compute the covariant derivative of the tangent vector field to the curve $\gamma$ on the surface of the cylinder.

(b) Use this to compute the geodesic curvature of $\gamma$ (use whichever orientation of the cylinder you prefer).

(c) Find a plane curve $\mathbf{y}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$ (in $\mathbb{R}^2$) whose curvature is the same as the geodesic curvature of $\gamma$?

2. (a) Similarly to question 1, show that the geodesic curvature of the curve $\gamma(t) = \mathbf{x}(z(t))$ where $z(t) = (\cos t, \sin t)$ on the surface of the cylinder is 1.

(b) Given any curve $z(t) = (u(t), \phi(t))$, show by calculation that the plane curvature of $z(t)$ is equal to the geodesic curvature of $\gamma(t) = \mathbf{x}(z(t))$ up to a sign determined by the orientation of the cylinder.

(c) Give a fast geometric argument for why part (b) is true and why it will not be true for curves on a general surface.

3. Directly calculate the integral of the 1-form $\alpha = xy \, dy$ over the unit circle in $\mathbb{R}^2$. Why is the answer obvious using Stokes’ theorem?

4. Suppose $\Sigma$ is a regular surface with boundary $\partial \Sigma$ and that $\Sigma$ has no self-intersections. Suppose also that $\beta = 2xy \, dx \wedge dy + 2xz \, dx \wedge dz$ is a 2-form on $\mathbb{R}^3$. Show that the integral of $\beta$ over $\Sigma$ depends only on the boundary $\partial \Sigma$. Calculate $\int_\Sigma \beta$ when $\partial \Sigma$ is the circle $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, x - z = 0\}$.

5. (Harder - do question 7 on sheet 6 first) Use the integral of the 1-form $\alpha = x \, dy$ to calculate the area of a regular $n$-gon whose vertices lie on a circle of radius 1. Compute the side length of a regular 17-gon of area 1. Hint: Let the corners of the $n$-gon have co-ordinates $c_k = (\cos \frac{2\pi}{n}k, \sin \frac{2\pi}{n}k)$ for $k = 0, \ldots, n - 1$. First calculate $\alpha$ restricted to the $k$th edge joining $c_k, c_{k+1}$. 

1