Math 162B: Differential Geometry Homework 6

Hand in questions 4, 7 & 11 at lecture Monday 9th March. The questions cover everything up to the end of the course — this will be the closest to a mock final you get!

1. Prove first that if $\gamma(t)$ is a curve whose speed is non-unit, then the geodesic curvature of γ is given by
$$\kappa_g \mathbf{U} = \frac{\gamma' \times D_{\frac{\mathrm{d}}{\mathrm{d}t}} \gamma'}{|\gamma'|^3}.$$
(Just as when we calculated the curvature of a variable speed spacecurve in 162A, reparameterize $\mu(s) = \gamma(t(s))$ to be unit speed and calculate). Secondly prove that we need not calculate the covariant derivative directly, that
$$\kappa_g = \frac{(\gamma' \times \gamma'') \cdot \mathbf{U}}{|\gamma'|^3}.$$

2. Consider the vertical cylinder of radius 1 ($x^2 + y^2 = 1$) oriented with outward pointing normal, and its intersection with the plane $z = x \tan \psi$ for fixed ψ.
   (a) By considering the co-ordinates $\hat{x} = \sqrt{x^2 + z^2}$ and $\hat{y} = y$ in the plane $z = x \tan \psi$, show that the intersection is an ellipse.
   (b) Calculate the curvature, geodesic curvature and normal curvature of this ellipse (it's probably easiest to parameterize everything in terms of the rotational angle ϕ around the cylinder).

3. Check that r, θ are oriented co-ordinates for the plane (away from the origin). I.e. check that $\mathrm{d}r \wedge \mathrm{d}\theta$ is a positive multiple of $\mathrm{d}x \wedge \mathrm{d}y$.

4. Let
$$x(\theta, \phi) = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$
be the usual polar co-ordinate map for the unit sphere. Calculate $x^*(\mathrm{d}x_1 \wedge \mathrm{d}x_2)$ hence show that the integral of $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ over the unit sphere is zero.

5. What do you get if you integrate
$$\alpha = x_1 \,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + x_2 \,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_3 \,\mathrm{d}x_1 \wedge \mathrm{d}x_2$$
over a region of the sphere of radius a (use polar co-ordinates).

6. Write out the proof of Stokes' theorem for a cubic region Σ in $\mathbb{R}^3$.

7. Taking the inside of a simple closed curve γ in the plane with its usual orientation, show that the induced orientation corresponds to traversing γ counter-clockwise. What does $\int_\gamma x \,\mathrm{d}y$ calculate?

8. Take the inside of the unit sphere with usual orientation on $\mathbb{R}^3$ - show that the induced orientation on the surface S of the sphere corresponds to taking the outward pointing normal. Calculate $\int_S x_1 \, dx_2 \wedge dx_3$.

9. Extending question 5, verify Stokes' theorem for the inside of the unit sphere bounded by the sphere of radius a where

$$\alpha = x_1 \, dx_2 \wedge dx_3 + x_2 \, dx_3 \wedge dx_1 + x_3 \, dx_1 \wedge dx_2.$$

10. Integrate the function $f(\mathbf{x}) = z$ over the unit sphere in $\mathbb{E}^3$.

11. What is the area of the region of the hyperbolic upper half plane given by $a < x < b$ and $p < y < q$ where p, q are both positive? Show that the area becomes infinite as $p \to 0$.

12. We have already seen that the surface of revolution

$$\mathbf{x}(u, \phi) = \begin{pmatrix} \cosh u \cos \phi \\ \cosh u \sin \phi \\ u \end{pmatrix}$$

is a minimal surface. Calculate the area of this surface between $u = -1$ and $u = 1$ and show that it is less than the area of the cylinder whose boundary is the same two circles.

13. Let $a > 0$ and calculate the area of the region of the hyperbolic upper half plane bounded by the three geodesics $(x \pm a/2)^2 + y^2 = a^2/4$ and $x^2 + y^2 = a^2$.

14. Show that for a geodesic pentagon, the sum of whose five angles is A, we have

$$\int K\theta_1 \wedge \theta_2 = A - 3\pi.$$

15. Consider the surface of revolution of a function $f(u)$ between the points $u = a$ and $u = b$. Suppose that f has a local minimum at a and a local maximum at b. By considering a geodesic quadrilateral consisting of half of the lines of latitude at $u = a$ and $u = b$ and the lines of longditude joining their ends, show that the integral of K over the region of the surface between $u = a$ and $u = b$ is zero.

16. What (roughly) is the percentage error in assuming that for a geodesic triangle the area of California (163696 square miles) on the Earth's surface, the sum of the angles is π? (The radius of the Earth is approximately 3959 miles).

17. Suppose that $\mathbf{x}(z(t))$ is a geodesic in the surface $\mathbf{x}$, and that $\mathbf{x}' = \cos \psi \mathbf{e}_1 + \sin \psi \mathbf{e}_2$. Use the first geodesic equation (rather than the proof in the notes with $\kappa_g = 0$) to prove directly that $\psi'(t) = \omega_{12}(z')$.

18. Calculate the Euler characteristic of the sphere by considering the dissections corresponding to the tetrahedron, octahedron and cube. Check that in each case you get the same answer.

19. Show that if there is a regular solid whose faces are pentagons and such that 3 meet at each vertex, then there must be 12 faces. How many faces must there be if the faces are equilateral triangles such that 5 meet at each vertex?

20. A soccer ball is made by sewing together regular pentagons and hexagons, with three pieces meeting at each vertex. How many pentagons are there? If you assume also that one pentagon and two hexagons meet at each vertex, how many hexagons are there?

21. Let Σ be a surface obtained from a sphere by deforming it smoothly. Show that there exists a point on Σ where $K > 0$.

22. Given a local surface $\mathbf{x} : U \to \mathbb{E}^3$, regard the unit normal $\mathbf{e}_3$ as defining a parameterization of the unit sphere (we call $\mathbf{e}_3 : U \to S^2$ the *Gauss map* of $\mathbf{x}$). Show that

$$d\mathbf{e}_3 \overset{\wedge}{\times} d\mathbf{e}_3 = 2\omega_{13} \wedge \omega_{23}\mathbf{e}_3,$$

and hence deduce that the area form for the Gauss map is $K\theta_1 \wedge \theta_2$, where K is the Gauss curvature of the original surface. Hence show that if the Gauss map of an oriented, bounded surface Σ without boundary is a bijection, then Σ has Euler characteristic 2.

23. Calculate the Euler characteristic of a disc by observing that it is topologically identical to the upper half of a sphere (conveniently the edge is a geodesic) and using the Gauss–Bonnet theorem.

24. Prove that the area contained inside a circle on a sphere of radius R, the circle's circumference, and its (constant) geodesic curvature are related by

$$\kappa_g = \frac{2\pi R^2 - \text{Area}}{R^2 \cdot \text{Circumference}}.$$

25. Consider a polygon in the plane. To what basic result does the full Gauss–Bonnet theorem reduce?

26. State the relationship between the Gauss curvature of a surface and the sum of the angles in a geodesic triangle. Suppose that a surface has $K < -1$ everywhere. Prove that the area of any geodesic triangle is less than π.

27. Draw a triangular dissection of a finite length circular cylinder in order to calculate its Euler characteristic. Verify that the Gauss–Bonnet theorem holds.

28. Consider the torus of revolution parameterized by

$$\mathbf{x}(\psi, \phi) = \begin{pmatrix} (a + r\sin\psi)\cos\phi \\ (a + r\sin\psi)\sin\phi \\ r\cos\psi \end{pmatrix}, \qquad 0 \le \psi, \phi < 2\pi,$$

where $a > r > 0$ are constants. Find K and show explicitly that the Gauss–Bonnet theorem holds for this surface.

29. Let Σ be a simply connected subset of the plane from which a single circular hole has been removed. Construct a simple dissection of Σ into 2 faces to show that its Euler characteristic is 0. Generalize the construction to show that a simply connected set from which g holes have been removed has $\chi = 1 - g$.

30. What is the Euler characteristic of a torus from which a single small circular patch has been removed? Explain.