Math 162B: Differential Geometry Midterm

Marks per question are in brackets: Total = 50

The following formulae for an adaptive frame are provided for reference.

$$
\begin{aligned}
&\mathrm{d}\theta_1 + \omega_{12} \wedge \theta_2 = 0 && \mathrm{d}\omega_{12} + \omega_{13} \wedge \omega_{32} = 0 \\
&\mathrm{d}\theta_2 + \omega_{21} \wedge \theta_1 = 0 && \mathrm{d}\omega_{13} + \omega_{12} \wedge \omega_{23} = 0 \\
&\theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23} = 0 && \mathrm{d}\omega_{23} + \omega_{21} \wedge \omega_{13} = 0
\end{aligned}
$$

$$
\mathrm{I} = \theta_1^2 + \theta_2^2, \qquad \mathrm{I\!I} = -\theta_1\omega_{13} - \theta_2\omega_{23}.
$$

U always denotes a domain in $\mathbb{R}^2$.

1. Let $\alpha = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$ in $\mathbb{R}^4$. Calculate $\alpha \wedge \alpha$. (5)

2. (a) Write down the relation between a moving frame $\mathbf{E}$, its differential, and its *matrix* of connection 1-forms ω. (3)

   (b) Let $\mathbf{E}, \hat{\mathbf{E}}$ be moving frames related by $\hat{\mathbf{E}} = \mathbf{E}A$ (so that $A : U \to \mathrm{SO}(3)$ is a smooth map into the space of 3×3 orthogonal matrices with $\det A = 1$). Prove that the connection matrix $\hat{\omega}$ of $\hat{\mathbf{E}}$ is related to the connection matrix ω of $\mathbf{E}$ by $\hat{\omega} = A^{-1}\omega A + A^{-1}\,\mathrm{d}A$. (7)

3. Let $f(x)$ be a smooth function on U, and consider the abstract metric $\mathrm{I} = (f(x))^{-2}\,\mathrm{d}x^2 + f(x)^2\,\mathrm{d}y^2$ defined wherever $f(x) \neq 0$. Show that the Gauss curvature of I exists everywhere, even where $f(x) = 0$. In partiular, calculate the Gauss curvature when $f(x) = 1 - x^{-1}$ at $x = 1$, and explain what happens as $x \to 0$. (15)

4. Recall that if you are given connection 1-forms ω_{ij} and 1-forms θ_1, θ_2 that together satsify the structure equations for a surface, then, up to rigid motions, there exists a unique surface $\mathbf{x}$ and adaptive frame $\mathbf{E}$ with these forms. You are given the abstract first fundamental form $\mathrm{I} = \mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$, and told that its Gauss curvature is $K = 1$.

   (a) Suppose that $\mathbf{x}$ is a surface with given I, and whose second fundamental form is diagonal in these co-ordinates. Argue that $\mathrm{I\!I} = a^{-1}\,\mathrm{d}u^2 + a\cos^2 u\,\mathrm{d}v^2$ for some function a (5)

   (b) Suppose that the function a is constant.

      i. Write down $\theta_1, \theta_2, \omega_{13}, \omega_{23}$ and find ω_{12}. (5)

      ii. Use the structure equations to show that the only values of a for which there exists a surface with $\mathrm{I}, \mathrm{I\!I}$ as given, are $a = \pm 1$. (7)

      iii. Identify these surfaces (you may quote theorems of the class). (3)