Math 162B: Differential Geometry Midterm

Marks per question are in brackets: Total = 50

The following formulae for an adaptive frame are provided for reference.

\[
\begin{align*}
\theta_1 + \omega_{12} \wedge \theta_2 &= 0 \\
\theta_2 + \omega_{21} \wedge \theta_1 &= 0 \\
\theta_1 \wedge \omega_{13} + \theta_2 \wedge \omega_{23} &= 0 \\
I &= \theta_1^2 + \theta_2^2, \quad \II = -\theta_1 \omega_{13} - \theta_2 \omega_{23}.
\end{align*}
\]

\(U\) always denotes a domain in \(\mathbb{R}^2\).

1. Let \(\alpha = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge d\theta_4\) in \(\mathbb{R}^4\). Calculate \(\alpha \wedge \alpha\). (5)

2. (a) Write down the relation between a moving frame \(E\), its differential, and its matrix of connection 1-forms \(\omega\). (3)

   (b) Let \(E, \hat{E}\) be moving frames related by \(\hat{E} = EA\) (so that \(A : U \to SO(3)\) is a smooth map into the space of \(3 \times 3\) orthogonal matrices with \(\det A = 1\)). Prove that the connection matrix \(\hat{\omega}\) of \(\hat{E}\) is related to the connection matrix \(\omega\) of \(E\) by \(\hat{\omega} = A^{-1} \omega A + A^{-1} dA\). (7)

3. Let \(f(x)\) be a smooth function on \(U\), and consider the abstract metric \(I = (f(x))^{-2} dx^2 + f(x)^2 dy^2\) defined wherever \(f(x) \neq 0\). Show that the Gauss curvature of \(I\) exists everywhere, even where \(f(x) = 0\). In particular, calculate the Gauss curvature when \(f(x) = 1 - x^{-1}\) at \(x = 1\), and explain what happens as \(x \to 0\). (15)

4. Recall that if you are given connection 1-forms \(\omega_{ij}\) and 1-forms \(\theta_1, \theta_2\) that together satisfy the structure equations for a surface, then, up to rigid motions, there exists a unique surface \(x\) and adaptive frame \(E\) with these forms. You are given the abstract first fundamental form \(I = du^2 + \cos^2 u dv^2\), and told that its Gauss curvature is \(K = 1\).

   (a) Suppose that \(x\) is a surface with given \(I\), and whose second fundamental form is diagonal in these co-ordinates. Argue that \(\II = a^{-1} du^2 + a \cos^2 u dv^2\) for some function \(a\) (5)

   (b) Suppose that the function \(a\) is constant.

      i. Write down \(\theta_1, \theta_2, \omega_{13}, \omega_{23}\) and find \(\omega_{12}\). (5)

      ii. Use the structure equations to show that the only values of \(a\) for which there exists a surface with \(I, \II\) as given, are \(a = \pm 1\). (7)

      iii. Identify these surfaces (you may quote theorems of the class). (3)