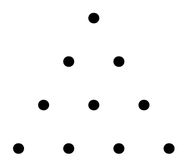
Math 180A: Homework 1

Submit the * questions by Friday 14^{th} January

1. In this question we think about the richness of *square-triangular numbers.* A number is *triangular* if it is the sum of the first n natural numbers. For example:

$$1 = 1$$
$$3 = 1 + 2$$
$$6 = 1 + 2 + 3$$
$$10 = 1 + 2 + 3 + 4$$



   (a) Prove that a number is triangular if and only if it may be written in the form $\frac{1}{2}n(n+1)$ where n is a natural number.

   (b) The *square numbers* are those natural numbers of the form m^2. A number is *square-triangular* if it is both square and triangular. Certainly 1 is square-triangular. Find the next square-triangular number.

   (c) To find all square-triangular numbers m^2 is equivalent to finding all integer solutions (m, n) to the equation

$$m^2 = \frac{1}{2}n(n+1)$$

   One way to proceed is as follows: certainly any n satisfying this equation is either even or odd. Prove that finding a square-triangular number is equivalent to being able to find integers (m, k) which solve either of the equations

$$m^2 = k(2k+1) \quad \text{or} \quad m^2 = k(2k-1)$$

   (d) Suppose that $d \in \mathbb{N}$ is a divisor of *both* k and $2k+1$. Prove that $d = 1$.

   (e) It follows from part (d) that if (m, k) solves $m^2 = k(2k+1)$, then *both* k and $2k+1$ must be *perfect squares.*
   Prove that finding square-triangular numbers is equivalent to finding all integer solutions (x, y) to the equations

$$x^2 - 2y^2 = \pm 1$$

   (f) Using a spreadsheet or by writing a computer program, you should be able to find the first few pairs of solutions (x, y) to these equations. Hence find the first *five* square-triangular numbers.
   (*Hint: try computing* $\sqrt{2y^2 \pm 1}$ *for* $y = 1, 2, 3, 4, \ldots$ *and spotting when this is an integer...*)

The equation $x^2 - 2y^2 = 1$ is an example of *Pell's equation.* The solutions of this famous equation are related to all manner of fun things such as *continued fractions* and rational approximations to $\sqrt{2}$. For example $(99, 70)$ is a solution and $\frac{99}{70} = 1.4142857\ldots \approx \sqrt{2}$. We shall see how to find the *infinitely many* solutions to Pell's equation in Math 180B, and hence find all the square-triangular numbers.

2. * Try this alternative approach to finding all primitive Pythagorean triples (x, y, z) where y is even. Let $\hat{y} = \frac{1}{2}y$. Then

$$\hat{y}^2 = \frac{1}{4}y^2 = \frac{1}{4}(z^2 - x^2) = \frac{z-x}{2} \cdot \frac{z+x}{2}.$$

This RHS is the product of two integers, since x, z are both odd.

(a) Explain why $\frac{z-x}{2}$ and $\frac{z+x}{2}$ have no common factors.

Continuing the argument: $\hat{y}^2$ is a perfect square, and since they are relatively prime, $\frac{z-x}{2}, \frac{z+x}{2}$ must also be perfect squares. Define positive integers u, v by[1]

$$u^2 = \frac{1}{2}(z+x), \qquad v^2 = \frac{1}{2}(z-x).$$

(b) Find x, y and z in terms of u and v.

(c) Argue that u and v have no common factor and that both cannot be odd.

(d) Compare with the solution given in lectures: what are s and t in terms of u and v.

3. Try adding up the first few odd numbers and see if the numbers you get satisfy some sort of pattern. Once you find the pattern, express it as a formula. Can you find a geometric verificiation that your formula is correct?
(*Hint: How might you create a square with $n+1$ dots per side from a square with n dots per side?*)

4. * The consecutive odd numbers 3, 5, and 7 are all primes. Are there infinitely many such "prime triplets"? That is, are there infinitely many prime numbers p such that $p+2$ and $p+4$ are also prime?

5. * (a) We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.

(b) By examining a list of primitive Pythagorean triples, make a guess about when a, b or c is a multiple of 5. Try to show that your guess is correct.

6. * Let m and n be positive integers that differ by 2, and write the sum $\frac{1}{m} + \frac{1}{n}$ as a fraction in lowest terms. For example $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$.

(a) Compute the next three examples.

(b) Examine the numerators and denominators of the fractions in (a) and compare them with a table of primitive Pythagorean triples. Formulate a conjecture about such fractions.

(c) Prove that your conjecture is correct.
(*Hint: $m-1$ and $m+1$ differ by 2...*)

7. * (a) Use the lines through the point $(1, 1)$ to describe all the points of the circle $x^2 + y^2 = 2$ whose co-ordinates are rational numbers.

(b) What goes wrong if you try to apply the same procedure to find all the points on the circle $x^2 + y^2 = 3$ with rational co-ordinates?

[1] Note that $z > x$ automatically so $v^2 > 0$.

8. (a) Consider a general cubic polynomial equation

$$(x-a)(x-b)(x-c) = x^3 + p_2 x^2 + p_1 x + p_0 = 0$$

where a, b, c are the roots. Prove that if the coefficients $p_0, \ldots, p_2$ are rational numbers and that *two* of the roots are rational, then so is the third root.

(b) The curve $y^2 = x^3 + 8$ contains the points $(1, -3)$ and $(-\frac{7}{4}, \frac{13}{8})$. The line through these two points intersects the curve in exactly one other point. Find it.
Your calculation in part (a) should help.