Math 180A: Homework 2

Submit the * questions at the discussion on Thursday 25th January
(Use a calculator if to help with the Euclidean Algorithm calculations!)

1. * Verify the following elementary properties of divisibility, where $a, b, c$ are integers.
   
   (a) $a | 0, a | a$, and $±1 | a$.
   (b) If $a | b$ and $b | c$, then $a | c$ (divides is transitive).
   (c) If $a | b$ and $a | c$, then $a | (bx + cy)$ for all $x, y \in \mathbb{Z}$.

2. Use the Euclidean Algorithm to compute the following gcd's.

   (a)* gcd$(121, 105)$  (b)* gcd$(12345, 67890)$  (c) gcd$(54321, 9876)$

3. Evaluate gcd$(4655, 12075)$ and express the result as a linear combination of 4655 and 12075 with coefficients in $\mathbb{Z}$.

4. Find all the integer solutions (if any exist) to the following equations (for (f) you will need to use the method several times):

   (a) * $4x - y = 7$  (b) * $12x + 4y = 10$  (c) * $105x - 121y = 1$

   (d) $2072x + 1813y = 2849$  (e) * $12345x - 67890y = \gcd(12345, 67890)$

   (f) \[
   \begin{cases}
   7x + 2y = 21 \\
   3x - 7z = 2
   \end{cases}
   \]

5. * Find all solutions of $19x + 20y = 1909$ with $x > 0$ and $y > 0$.

6. Let $r_0, r_1, r_2, \ldots$ be the successive remainders in the Euclidean Algorithm applied to $a > b > 0$ (we take $b = r_0$). Show that after every two steps, the remainder is reduced by at least one half.

   In other words, verify that $r_{i+2} < \frac{1}{2} r_i \quad \forall i = 0, 1, 2, 3 \ldots$

   Conclude that the Euclidean Algorithm terminates in at most $2 \log_2 b$ steps. In particular, show that the number of steps is at most seven times the number of digits in $b$.

7. The sequence of Fibonacci numbers $(F_n)_{n=1}^{\infty} = (1, 1, 2, 3, 5, 8, 13, \ldots)$ is defined by the recurrence relation

   \[
   \begin{cases}
   F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}, \\
   F_1 = F_2 = 1
   \end{cases}
   \]

   (a) Prove that no two successive Fibonacci numbers have a common divisor $a > 1$.

   (b) Apply the Euclidean Algorithm to find $\gcd(F_7, F_6) = \gcd(13, 8)$. Repeat for $\gcd(F_8, F_7) = \gcd(21, 13)$. 

(c) Make a hypothesis about how many steps are necessary in order for the Euclidean Algorithm to compute \( \gcd(F_{n+1}, F_n) \).

(d) Compute \( 2 \log_2 F_n \) for \( n = 4, 5, 6, 7, 8 \). Thinking about the conclusion of the previous question, why might we say that the Euclidean Algorithm is very slow when applied to successive Fibonacci numbers?

8. * Let \( a, b, r, s \) be given constants. Prove that the arithmetic progressions

\[
\{ax + r : x \in \mathbb{Z}\} \quad \text{and} \quad \{by + s : y \in \mathbb{Z}\}
\]

intersect if and only if \( \gcd(a, b) \mid (s - r) \).

9. Show that if \( ad - bc = \pm 1 \), then the fraction

\[
\frac{a + b}{c + d}
\]

is in reduced form (i.e. \( \gcd(a + b, c + d) = 1 \)).

10. * Show that if \( \gcd(a, b) = 1 \), then \( \gcd(a - b, a + b) = 1 \) or 2. Exactly when is the value 2?