Math 180B - Notes

Neil Donaldson

Spring 2021

1 Continued Fractions

We consider a generalization of the Euclidean Algorithm which has ancient historical roots and yet still has relevance and applications today.

1.1 Continued Fraction Representations of Rational Numbers

We start with a simple example, messing with the fraction $\frac{61}{14}$. First write the fraction as an integer plus a proper fraction:

$$\frac{61}{14} = 4 + \frac{5}{14}$$

Now flip the second fraction upside down and repeat until we can go no further:

$$\frac{61}{14} = 4 + \frac{5}{14} = 4 + \frac{1}{\frac{14}{5}} = 4 + \frac{1}{2+\frac{4}{5}} = 4 + \cfrac{1}{2+\cfrac{1}{\frac{5}{4}}} = 4 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4}}}$$

We stop because the final fraction $\frac{1}{4}$ is the reciprocal of an integer. We call this representation a *continued fraction*. Rather than write the whole thing out, we use a simpler notation:

$$\frac{61}{14} = [4; 2, 1, 4]$$

Now for the promised tie-in with the Euclidean Algorithm: to compute $\gcd(61, 14)$;

$$\begin{aligned}
\underline{61} &= 4 \cdot \underline{14} + \underline{5} \\
\underline{14} &= 2 \cdot \underline{5} + \underline{4} \\
\underline{5} &= 1 \cdot \underline{4} + \underline{1} \\
\underline{4} &= 4 \cdot \underline{1}
\end{aligned}$$

The digits of the continued fraction are precisely the sequence of *quotients*[1] from the Euclidean Algorithm.

[1] Recall that at each stage we apply the Division Algorithm: if $a > b > 0$ then there exist unique integers, the *quotient* q and *remainder* r which satisfy

$$a = qb + r \quad \text{and} \quad 0 \le r < b$$

What's the Point? Why might we want such a convoluted expression for a fraction? One answer is that by truncating a continued fraction we obtain *approximations* to our original fraction. For example:

$$[4] = 4$$

$$[4;2] = 4 + \frac{1}{2} = \frac{9}{2} = 4.5$$

$$[4;2,1] = 4 + \frac{1}{2+\frac{1}{1}} = 4\frac{1}{3} = \frac{13}{3} = 4.333\ldots$$

$$[4;2,1,4] = 4 + \frac{1}{2+\frac{1}{\frac{5}{4}}} = 4 + \frac{1}{2+\cfrac{1}{1+\cfrac{1}{4}}} = \frac{61}{14} = 4.35714\ldots$$

These approximations alternate below and above the final fraction, becoming successively closer. In fact, in a certain sense, $\frac{13}{3}$ is the best rational approximation to $\frac{61}{14}$ among all fractions with denominator at most 13 (see Exercise 4). Finding rational approximations with small denominators was crucial in the days before calculators.

We shall shortly prove that the coefficients of a continued fraction are always the quotients from the Euclidean Algorithm: try to do this yourself before seeing the proof in a few pages. Before being more formal, we make a historical digression to consider one of the oldest discussions of irrationality, for it is in the realm of irrational numbers that continued fractions really show their worth.

Exercises 1. Compute the continued fraction representation of $\frac{105}{39}$ using the direct method in this section.

2. Evaluate the rational number with continued fraction $[2;1,5,1,2]$.

3. If $x = [c_1; c_2, \ldots, c_n]$, find the continued fraction for $\frac{1}{x}$.

4. Investigate the claim that $\frac{13}{3}$ is a good rational approximation to $\frac{61}{14}$ among all fractions whose denominator is less than 14. Specifically, for each $q \in \{1, \ldots, 13\}$ find the integer p for which $\left|\frac{p}{q} - \frac{61}{14}\right|$ is minimal and then compute $\left|p - \frac{61}{14}q\right|$, the error weighted by the size of the denominator. What do you notice?
(*If you're feeling adventurous, automate this using a computer*)

5. Among all fractions with denominator at most 202, what do you think is the best rational approximation to $\frac{254}{203}$ weighted by the size of the denominator? Why?

1.2 Incommensurability and the Ancient Greeks (non-examinable)

The ancient Greek mathematician Theaetetus (417–369 BC) developed a notion of irrationality which formed the basis of the longest and most difficult book (Book X) of Euclid's *Elements* (c. 300 BC). Irrationality presented a great philosophical challenge to the Greeks since they only accorded positive integers the status of *number.* These would then be used to *compare* lengths by menas of ratios. For instance, two rods might be described as having lengths in the ratio 17 : 5. To us, of course, this is really the ratioal number $\frac{5}{17}$ in disguise. The problem for the Greeks was that some lengths are not in integer ratios.

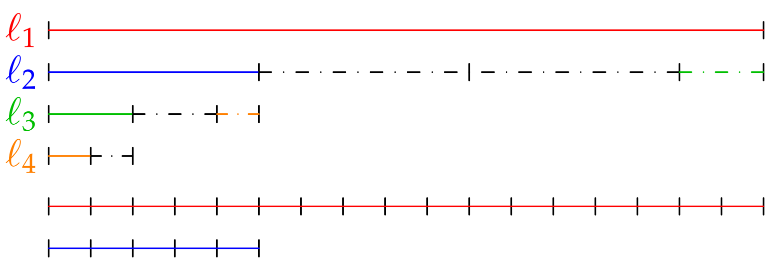
Two lengths/rods ℓ_1 and ℓ_2 were considered *commensurable* if there was some sub-length m and positive integers a, b such that $\ell_1 = am$ and $\ell_2 = bm$. Otherwise said, the lengths are in proportion

$$\ell_1 : \ell_2 = a : b$$

In modern times, we happily equate the concepts of length and numberand say that the ratio $\frac{\ell_1}{\ell_2} = \frac{a}{b}$ is rational. To the Greeks however, there remained some questions:

1. If two lengths are commensurable,[2] how do we find a common sub-length?
2. If two lengths are incommensurable, how do we show that no common sub-length exists?

Theaetetus answer to these questions is essentially the Euclidean Algorithm. If $\ell_1 : \ell_2$ is in an integer ratio, then the greatest common divisor of ℓ_1 and ℓ_2 provides a suitable common sublength. Here is an example with two initial lengths ℓ_1 and ℓ_2:



$$\ell_1 = 3\ell_2 + \ell_3$$
$$\ell_2 = 2\ell_3 + \ell_4$$
$$\ell_3 = 2\ell_4$$
$$\ell_1 = 17\ell_4$$
$$\ell_2 = 5\ell_4$$

In this case, ℓ_4 is a common sub-length. By choosing units so that $\ell_4 = 1$ we see that we've applied the Euclidean Algorithm to calculate $\gcd(17, 5) = 1$.

The point is that the Euclidean Algorithm works perfectly well when applied to *lengths*, not just integers. Indeed, to the Greeks, the Division Algorithm was interpreted thus; if $\ell_1 > \ell_2$ are rods of positive length, there exists a unique integer $q \geq 1$ and a length r for which,

1. ℓ_1 may be measured by q copies of ℓ_2 plus a single copy of r: $\quad \ell_1 = q\ell_2 + r$.
2. r is shorter than ℓ_2: $\quad 0 \leq r < \ell_2$.

We can now give Theaetetus' definition of commensurability.

Definition 1.1. Lengths ℓ_1, ℓ_2 are *commensurable* if the Euclidean Algorithm terminates, in which case a common sub-length is the last length in the sequence.

Lengths are *incommensurable* if the Algorithm never terminates.

In modern language, ℓ_1, ℓ_2 are commensurable $\iff \frac{\ell_1}{\ell_2} \in \mathbb{Q}$.

Here is an example of a simple result in this language.

Theorem 1.2. *Suppose that lengths ℓ, m are in the golden ratio; that is $\ell + m : \ell = \ell : m$. Then ℓ and m are incommensurable. Otherwise said, the golden ratio is irrational.*

[2] In the days of Pythagoras (c.590–495 BC) many falsely assumed that *all* lengths were commensurable. The understanding that the side and diagonal of a square are incommensurable (i.e. $\sqrt{2} \notin \mathbb{Q}$) caused a crisis of faith for the Pythagoreans.

Proof. Clearly[a] $\frac{\ell}{m} = \frac{\ell+m}{\ell} > 1$ since $m > 0$. However, if $\ell \geq 2m$ we obtain a contradiction

$$2 \leq \frac{\ell}{m} = \frac{\ell+m}{\ell} = \frac{2\ell+2m}{2\ell} \leq \frac{3\ell}{2\ell} = \frac{3}{2} \tag{$*$}$$

It follows that the first quotient in the algorithm is 1 and that the first line reads

$$\ell = 1 \cdot m + (\ell - m)$$

Now continue;

$$\frac{\ell - m}{m} = \frac{\ell}{m} - 1 = \frac{m}{\ell} \tag{$\dagger$}$$

says that the ratio $\frac{m}{\ell-m}$ is also golden! The second line of the algorithm therefore has quotient 1, as do *all* subsequent lines: the algorithm continues *ad infinitum*, with all quotients equalling 1. ∎

[a] For clarity, we write in fractions so that the golden ratio reads $\frac{\ell+m}{\ell} = \frac{\ell}{m}$. This would have made no sense to the Greeks, since length and number were distinct concepts. To be more historically accurate, we should write ($*$) as

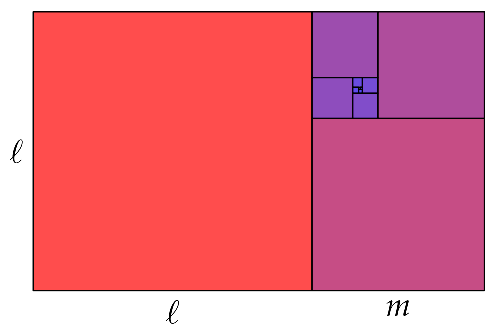
$$2:1 \leq \ell : m = \ell + m : \ell = 2\ell + 2m : 2\ell \leq 3\ell : 2\ell = 3:2$$

and ($\dagger$) using subtraction of ratios

$$\begin{cases} \ell + m : \ell = \ell : m \\ \ell : \ell = m : m \end{cases} \implies m : \ell = \ell - m : m$$

The Greeks did not have symbolic algebra, but this method of computing with ratios was well understood.

You've probably seen the famous picture: each step of the algorithm can be visualized as deleting a square from a golden rectangle, leaving a smaller (similar) golden rectangle.



The geometric approach of the Greeks is very difficult for us to follow. Indeed it is very hard to resist writing

$$\varphi = \frac{\ell}{m} \implies \varphi = 1 + \varphi^{-1} \implies \varphi^2 - \varphi - 1 = 0$$
$$\implies \varphi = \frac{\sqrt{5}+1}{2}$$

The Greeks, of course, had no notion of $\sqrt{5}$, so the above was essentially how they proceeded: fun!

Motivated by Theaetetus, we try a continued fraction approach for irrational numbers. If the golden ratio produces an infinite sequence of quotients equalling 1, is it reasonable to write the following?

$$\varphi = \frac{\sqrt{5}+1}{2} = [1;1,1,1,\ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

The answer will prove to be 'yes,' though the ellipses $\cdots$ present a difficulty: we are really making a claim about the *convergence* of some sequence to φ...

1.3 Continued Fractions and Irrational Numbers

We define the concept of a continued fraction for any real number.

Definition 1.3. Suppose $x \in \mathbb{R}$. Define a pair of sequences (c_n), (R_n) inductively

$$c_1 := \lfloor x \rfloor \qquad R_1 := x - c_1$$

$$c_n := \left\lfloor \frac{1}{R_{n-1}} \right\rfloor \qquad R_n := \frac{1}{R_{n-1}} - c_n$$

The notation $\lfloor x \rfloor$ is the *floor* of x: the greatest integer less than or equal to x. If any term R_n is zero, then the sequences terminate. The *continued fraction representation* of x is written $[c_1; c_2, c_3, \ldots]$.

Examples 1.4. 1. It is worth revisiting our first example in this language.

$$c_1 = \left\lfloor \frac{61}{14} \right\rfloor = 4 \qquad R_1 = \frac{61}{14} - 4 = \frac{5}{14}$$

$$c_2 = \left\lfloor \frac{14}{5} \right\rfloor = 2 \qquad R_2 = \frac{14}{5} - 2 = \frac{4}{5}$$

$$c_3 = \left\lfloor \frac{5}{4} \right\rfloor = 1 \qquad R_3 = \frac{5}{4} - 1 = \frac{1}{4}$$

$$c_4 = \lfloor 4 \rfloor = 4 \qquad R_4 = 4 - 4 = 0$$

This produces the same sequence and continued fraction representation we saw earlier.

2. Now we try this for the golden ratio.

If $\varphi = \frac{\sqrt{5}+1}{2}$, then $1 < \varphi < 2$, whence $c_1 = 1$ and $R_1 = \frac{\sqrt{5}-1}{2}$. Then

$$c_2 = \left\lfloor \frac{2}{\sqrt{5}-1} \right\rfloor = \left\lfloor \frac{\sqrt{5}+1}{2} \right\rfloor = 1 \implies R_2 = \frac{\sqrt{5}-1}{2}$$

Both sequences have already begun to repeat, hence we obtain the expected continued fraction

$$[1; 1, 1, 1, 1, 1, 1, \ldots]$$

3. Consider $x = \sqrt{13}$. Since $3 < \sqrt{13} < 4$ we have $c_1 = 3$ and $R_1 = \sqrt{13} - 3$. Thus

$$\frac{1}{R_1} = \frac{1}{\sqrt{13}-3} = \frac{\sqrt{13}+3}{4} = 1 + \frac{\sqrt{13}-1}{4} \implies c_2 = 1$$

Rinse and repeat:

n	1	2	3	4	5	6	7	8
c_n	3	1	1	1	1	6	1	1
R_n	$\sqrt{13}-3$	$\frac{\sqrt{13}-1}{4}$	$\frac{\sqrt{13}-2}{3}$	$\frac{\sqrt{13}-1}{3}$	$\frac{\sqrt{13}-3}{4}$	$\sqrt{13}-3$	$\frac{\sqrt{13}-1}{4}$	$\frac{\sqrt{13}-2}{3}$

Since the sequences (R_n), (c_n) have started to repeat, it follows that they will continue to do so. The continued fraction representation of $\sqrt{13}$ is therefore the repeating sequence

$$[3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \ldots]$$

4. The first few terms of the continued fraction expansion for π can be found similarly, though you'll need a computer/calculator. Unlike the previous examples, there is no ongoing pattern to find.

$$\begin{aligned}
c_1 &= \lfloor \pi \rfloor = 3 \implies R_1 = \pi - 3 \\
c_2 &= \left\lfloor \frac{1}{\pi-3} \right\rfloor = 7 \implies R_2 = \frac{1}{\pi-3} - 7 = \frac{22-7\pi}{\pi-3} && \text{(note the 22 and 7...)} \\
c_3 &= \left\lfloor \frac{\pi-3}{22-7\pi} \right\rfloor = 15 \implies R_3 = \frac{\pi-3}{22-7\pi} - 15 = \frac{106\pi-333}{22-7\pi} \\
c_4 &= \left\lfloor \frac{22-7\pi}{106\pi-333} \right\rfloor = 1 \implies R_4 = \cdots
\end{aligned}$$

We obtain the continued fraction $[3;7,15,1,\ldots]$.

These examples suggest the following:

Theorem 1.5. *Suppose $x \in \mathbb{R}$.*

1. *The continued fraction representation of x has finite length if and only if $x \in \mathbb{Q}$.*
2. *If $x = \frac{a}{b}$ is rational, then it equals its continued fraction representation which moreover consists of the sequence of quotients from the Euclidean Algorithm applied to the pair (a, b).*
3. *If x is any real number, its continued fraction representation is the sequence of quotients found by applying the Euclidean Algorithm (à la Theaetetus) to the pair $(x, 1)$.*

Before proving this, it is worth noting a couple of points. If $c_1 = x$, then x is an integer and its (very boring) continued fraction is itself. The first quotient c_1 in a continued fraction is negative if and only if x is negative: all the other quotients must be positive.

Proof. Suppose first that $x = \frac{a}{b} \in \mathbb{Q}^+$. Write the Euclidean Algorithm for finding $\gcd(a, b) = r_{n-1}$:

$$\begin{aligned}
a &= c_1 b + r_1 && 0 \le r_1 < b \\
b &= c_2 r_1 + r_2 && 0 \le r_2 < r_1 \\
r_1 &= c_3 r_2 + r_3 && 0 \le r_3 < r_2 \\
&\ \vdots \\
r_{n-3} &= c_{n-1} r_{n-2} + r_{n-1} && 0 \le r_{n-1} < r_{n-2} \\
r_{n-2} &= c_n r_{n-1}
\end{aligned}$$

Rearranging each line for the quotient, we obtain

$$c_k = \frac{r_{k-2}}{r_{k-1}} - \frac{r_k}{r_{k-1}} = \left\lfloor \frac{r_{k-2}}{r_{k-1}} \right\rfloor \tag{$*$}$$

since $0 \leq r_k < r_{k-1}$ and c_k is an integer. Now compute:

$$\frac{a}{b} = c_1 + \frac{r_1}{b} = c_1 + \frac{1}{\frac{b}{r_1}} = c_1 + \frac{1}{c_2 + \frac{r_2}{r_1}} = c_1 + \frac{1}{c_2 + \frac{1}{\frac{r_1}{r_2}}} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{r_3}{r_2}}}$$

$$= c_1 + \frac{1}{c_2 + \frac{1}{\cdots + \frac{r_{n-2}}{r_{n-1}}}} = c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{1}{\cdots + c_n}}} = [c_1; c_2, \ldots, c_n]$$

Thus any positive rational number $\frac{a}{b}$ has a finite continued fraction expansion whose coefficients are precisely the quotients in the Euclidean Algorithm. Conversely, if a continued fraction terminates, then it is certainly rational. We have therefore proved parts 1 and 2.

For part 3, $x - \lfloor x \rfloor$ is less than 1, and so

$$x = \lfloor x \rfloor \cdot 1 + (x - \lfloor x \rfloor) = c_1 \cdot 1 + R_1$$

is precisely the first line of Theaetetus' Algorithm applied to the pair $(x, 1)$. The next line is then

$$1 = \left\lfloor \frac{1}{R_1} \right\rfloor \cdot R_1 + \left(1 - \left\lfloor \frac{1}{R_1} \right\rfloor R_1\right) = c_2 \cdot R_1 + R_1 R_2$$

with positive remainder $R_1 R_2 = R_1 \left(\frac{1}{R_1} - \left\lfloor \frac{1}{R_1} \right\rfloor\right) < R_1$. And the third:

$$R_1 = \left\lfloor \frac{1}{R_2} \right\rfloor \cdot R_1 R_2 + R_1 \left(1 - \left\lfloor \frac{1}{R_2} \right\rfloor R_2\right) = c_3 \cdot R_1 R_2 + R_1 R_2 R_3$$

with positive remainder $R_1 R_2 R_3 = R_1 R_2 \left(\frac{1}{R_2} - \left\lfloor \frac{1}{R_2} \right\rfloor\right) < R_1 R_2$. Without being too formal (prove by induction if you like), we see that the k^{th} line has the form

$$R_1 \cdots R_{k-2} = c_k \cdot R_1 \cdots R_{k-1} + R_1 \cdots R_k$$

This proves 3: indeed the k^{th} remainder in the Algorithm is the product $R_1 \cdots R_k$.

■

Exercises 1. Compute the continued fraction representations of the following fractions using the method of this section.

(a) $\frac{25}{11}$ (b) $\frac{632}{13}$ (c) $\frac{13}{632}$

2. Find the continued fraction representations of $\sqrt{2}$ and $\sqrt{3}$.

3. Use a calculator/computer to find six terms of the continued fraction representation of e.

4. (Hard!) Give a more formal proof that $\frac{a}{b} = [c_1; \ldots, c_n]$ by induction.
(Hint: let $b = r_0$ and prove that $\frac{a}{b} = [c_1; c_2, \ldots, c_k + \frac{r_k}{r_{k+1}}]$ for all $k \leq n-1$: use $(*)$!)

1.4 Convergents of Continued Fractions

We've already seen that a continued fraction representation of a rational number is equal to that number. To see that the same is true for irrational numbers, we need a notion of convergence.

Definition 1.6. Given the sequence of quotients (c_n), define new sequences $(p_n), (q_n)$ via

$$\begin{aligned} p_0 &= 1 \qquad & p_1 &= c_1 \qquad & p_n &= c_n p_{n-1} + p_{n-2} \\ q_0 &= 0 \qquad & q_1 &= 1 \qquad & q_n &= c_n q_{n-1} + q_{n-2} \end{aligned}$$

The ratio $\frac{p_n}{q_n}$ is the n^{th} *convergent* of the continued fraction $[c_1; c_2, c_3, \ldots]$.

Example 1.7. For $x = \frac{61}{14}$, the sequences are

n	0	1	2	3	4
c_n		4	2	1	4
p_n	1	4	9	13	61
q_n	0	1	2	3	14

To help remember the formulæ, look at the pattern for how the red numbers produce the blue:

$$9 = 2 \cdot 4 + 1$$

Observe that the sequence of convergents

$$\left(\frac{p_n}{q_n}\right) = \left(4, \frac{9}{2}, \frac{13}{3}, \frac{61}{14}\right)$$

are precisely the approximations obtained on page 2 by truncating the continued fraction.

The same thing can be seen with any of our other examples: indeed we have the following result.

Theorem 1.8. $\dfrac{p_n}{q_n} = [c_1; \ldots, c_n]$

The result is easier to prove in the form of a two-part lemma.

Lemma 1.9. *Let $c_1, \ldots, c_n$ be integers such that the continued fraction $[c_1; \ldots, c_n]$ exists.*[a] *We can describe the continued fraction in terms of a matrix multiplication.*

1. $\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \implies [c_1; \ldots, c_n] = \frac{A}{C}.$

2. *In terms of the sequences (p_n) and (q_n), we have* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$.

[a] Strictly $c_2, \ldots, c_n$ are positive integers, and $c_n \geq 2$. Part 1 holds for any real numbers provided we never divide by zero: something like $[2; 1, -1] = 2 + \frac{1}{1 + \frac{1}{-1}}$ clearly cannot be allowed!

Proof. We prove part 1 by induction; part 2 is left as an exercise.
For the base case, observe that $[c_1] = c_1 = \frac{c_1}{1} = \frac{A}{C}$ when $\left(\begin{smallmatrix} c_1 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.
Now fix n and assume that any continued fraction of length $n-1$ satisfies the condition

$$[c_2; \ldots, c_n] = \frac{A}{C} \qquad \text{where} \qquad \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then

$$[c_1; c_2, \ldots, c_n] = c_1 + \frac{1}{[c_2; \ldots, c_n]} = c_1 + \frac{C}{A} = \frac{c_1 A + C}{A}$$

However

$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} c_1 A + C & c_1 B + D \\ A & B \end{pmatrix}$$

By induction, part 1 is proved. ∎

We now want to see that the convergents of an irrational number x really converge to x. This is a somewhat tedious piece of elementary analysis which starts by taking determinants of the statement of Lemma 1.9: since each matrix on the has determinant -1, we have

$$(-1)^n = p_n q_{n-1} - p_{n-1} q_n$$

Clearly $\gcd(p_n, q_n) = 1$, so the convergents are fractions in lowest terms. Moreover,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$$

from which we obtain a telescoping/alternating series:

$$c_1 + \sum_{k=2}^{n} \frac{(-1)^k}{q_k q_{k-1}} = \frac{p_1}{q_1} + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right) + \cdots + \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \frac{p_n}{q_n} = [c_1; c_2, \ldots, c_n]$$

Since (q_k) is an increasing sequence (for $k \geq 2$) it follows that the sequence $(\frac{1}{q_n q_{n-1}})$ decreases to zero. The sequence of convergents $\left(\frac{p_n}{q_n}\right)$ is therefore the n^{th} partial sum of a convergent alternating series, whose limit L satisfies

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \cdots < L < \cdots < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2} \qquad \text{and} \qquad \left| L - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

The sequence of convergents is indeed convergent! It remains to see that its limit is x. For this, first observe that for all $n \in \mathbb{N}$

$$x = c_1 + \cfrac{1}{c_2 + \cfrac{1}{\cdots + \cfrac{1}{c_n + R_n}}}$$

In particular, for the first three convergents,

$$x = c_1 + R_1 > c_1 = [c_1] = \frac{p_1}{q_1}$$

$$x = c_1 + \frac{1}{c_2 + R_2} < c_1 + \frac{1}{c_2} = [c_1; c_2] = \frac{p_2}{q_2}$$

$$x = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + R_3}} > c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3}} = [c_1; c_2, c_3] = \frac{p_3}{q_3}$$

The pattern is obvious; the convergents are alternately smaller (if n is odd) or larger (if n is even) than x. We therefore see that

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_5} < \frac{p_7}{q_7} < \cdots < x < \cdots < \frac{p_8}{q_8} < \frac{p_6}{q_6} < \frac{p_4}{q_4} < \frac{p_2}{q_2}$$

and we've (finally!) proved the first two parts of the following.

Theorem 1.10. *Suppose x is irrational and has continued fraction representation $[c_1; c_2, c_3, \ldots]$.*

1. *x is the limit of the sequence of convergents $\frac{p_n}{q_n} = [c_1; c_2, \ldots, c_n]$,*

$$x = \lim_{n \to \infty} \frac{p_n}{q_n} = c_1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{q_n q_{n-1}}$$

2. *The n^{th} convergent satisfies*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

3. *Each convergent is closer to x than the previous:*

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \left| x - \frac{p_n}{q_n} \right|$$

Part 3 is left as a tough exercise. With very slight modifications, the Theorem holds even when x is rational. Part 2 was particularly useful in the days before calculators to guarantee a desired level of accuracy in an approximation.

Given the Theorem, it is now legitimate for us to write

$$x = [c_1; c_2, c_3, \ldots]$$

for any real number x, where it is understood that this means the limit of the sequence of convergents when x is irrational. More theoretically, if you recall the discussion in analysis where the real numbers may be defined as the set of limits of sequences of rational numbers, this procedure produces a concrete example of a sequence which converges to a given irrational number.

Examples 1.11. 1. $\sqrt{13} = [3; 1, 1, 1, 1, 6, \ldots] = 3.605555$ to 5dp. Its convergents are

n	0	1	2	3	4	5	6	7	8	9	10
c_n		3	1	1	1	1	6	1	1	1	1
p_n	1	3	4	7	11	18	119	137	256	393	649
q_n	0	1	1	2	3	5	33	38	71	109	180
$\frac{p_n}{q_n}$		3	4	3.5	3.66667	3.6	3.60606	3.60526	3.60563	3.60550	3.60556

2. $\pi = [3; 7, 15, 1, 292, 1, \ldots]$ whence, to 8dp,

n	0	1	2	3	4	5	6
c_n		3	7	15	1	292	1
p_n	1	3	22	333	355	103993	104348
q_n	0	1	7	106	113	33102	33215
$\frac{p_n}{q_n}$		3	3.14285714	3.14150943	3.14159292	3.14159265	3.14159265

According to part 2 of the Theorem,

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{7 \cdot 106} = \frac{1}{742} = 0.0013477\ldots$$

In fact, $\left|\pi - \frac{22}{7}\right| = 0.0012644\ldots$, so the accuracy estimate is very good. Similarly

$$\left|\pi - \frac{333}{106}\right| < \frac{1}{106 \cdot 113} = \frac{1}{11978} = 0.0000834\ldots$$

Note that larger values in the sequence (c_n) make the denominators of the convergents for π increase faster and produce better approximations than those for $\sqrt{13}$. The golden ratio $\varphi = [1; 1, 1, 1, 1, \ldots]$ thus has the slowest converging sequence of convergents.

Exercises 1. Compute the first six convergents $(\frac{p_n}{q_n})$ of $\sqrt{2}$, $\sqrt{3}$ and e and find a rational approximation to e which is accurate to at least $\frac{1}{200}$.

2. Prove that $(q_n)_{n=2}^{\infty}$ is a strictly increasing sequence.

3. Suppose that an irrational number x can be written as two continued fractions:

$$x = [c_1; c_2, c_3, \cdots] = [d_1; d_2, d_3, \ldots]$$

Prove that $c_i = d_i$ for all i.

Is there any way for a *rational* number can have two distinct continued fraction representations? How?

4. (a) Prove part 2 of Lemma 1.9 by induction.
   (b) Prove the following. For any $n \geq 2$ and any number $y \neq -\frac{q_{n-2}}{q_{n-1}}$, we have

$$[c_1; \ldots, c_{n-1}, y] = \frac{p_{n-1}y + p_{n-2}}{q_{n-1}y + q_{n-2}}$$

This is a slight abuse of notation when $y \notin \mathbb{N}$, but it should still make sense.

5. (Very hard!) Prove part 3 of Theorem 1.10, that each convergent is closer to x than the previous:

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \left| x - \frac{p_n}{q_n} \right|$$

Hints: Use $q_{r+2} = c_{r+2}q_{r+1} + q_r \geq q_{r+1} + q_r$ *to prove that* $\frac{1}{q_{r+1}q_{r+2}} - \frac{1}{q_{r+2}q_{r+3}} < \frac{1}{q_r q_{r+1}} - \frac{1}{q_{r+1}q_{r+2}}$ *for all* r*. Now prove that*

$$x - \frac{p_n}{q_n} = (-1)^{n+1} \left(\frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1}q_{n+2}} + \frac{1}{q_{n+2}q_{n+3}} - \cdots \right)$$

and finally put it all together!

1.5 Periodicity of Continued Fractions

You've hopefully already guessed that irrational numbers containing a square root eventually have periodic continued fractions.

Example 1.12. Suppose $x = [2; 1, 4, 1, 4, 1, 4, \ldots]$. The eventually periodic terms can be written

$$y = [1; 4, 1, 4, \ldots] = [1; 4, y] = 1 + \cfrac{1}{4 + \cfrac{1}{y}}$$

Some simple algebra turns this into a quadratic equation:

$$4y^2 - 4y - 1 = 0 \implies y = \frac{1+\sqrt{2}}{2} \implies x = 2 + \frac{1}{y} = 2 + \frac{2}{1+\sqrt{2}} = 2\sqrt{2}$$

There is a general result here:

Theorem 1.13. *A continued fraction is ultimately periodic if and only if it is a* quadratic irrationality*: a number of the form* $\alpha + \sqrt{\beta}$ *where* $\alpha, \beta \in \mathbb{Q}$ *and* $\beta \neq 0$.

Sketch Proof. We only prove the $\implies$ direction, attributable to Euler. Suppose

$$x = [d_1; d_2, d_3, \ldots, d_r, c_1, c_2, \ldots, c_n, c_1, c_2, \ldots, c_n, \ldots] = [d_1; d_2, d_3, \ldots, d_r, y]$$

where $y = [c_1; c_2, \ldots, c_n, y]$. By Exercise 4, we see that

$$y = \frac{p_n y + p_{n-1}}{q_n y + q_{n-1}} \implies q_n y^2 + (q_{n-1} - p_n)y - p_{n-1} = 0$$

$$\implies y = \frac{p_n - q_{n-1} + \sqrt{(q_{n-1} - p_n)^2 + 4p_{n-1}q_n}}{2q_n} \qquad (y > 0!)$$

is a quadratic irrationality. Indeed y lies in the field[a] $\mathbb{Q}(\sqrt{\beta})$ where $\beta = (q_{n-1} - p_n)^2 + 4p_{n-1}q_n$. Since x is computed from y using only field operations, it follows that x also lies in $\mathbb{Q}(\sqrt{\beta})$. ∎

[a] $\mathbb{Q}(\sqrt{\beta}) = \{\delta + \epsilon\sqrt{\beta} : \delta, \epsilon \in \mathbb{Q}\}$ is an *extension field* of the rational numbers; it is closed under addition and multiplication, and all non-zero elements have multiplicative inverses.

The converse, courtesy of Lagrange, is slightly more difficult, but requires a lot more space! A more general result is also available, but it also requires more work:

Theorem 1.14. *Two continued fractions have the same eventually periodic sequence if and only if* $\exists a, b, c, d \in \mathbb{Z}$ *with* $|ad - bc| = 1$ *and* $y = \frac{ax+b}{cx+d}$.

Exercises

1. (a) Write the continued fraction representation of $\sqrt{2}$ in the form $\sqrt{2} = 1 + \frac{1}{y}$ and prove that $y^2 - 2y - 1 = 0$. Why are you not surprised by the solutions to this equation?

   (b) Recall that $\sqrt{13} = [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \ldots]$. Show that $\sqrt{13} = 3 + \frac{1}{y}$ where y also satisfies a quadratic equation. Solve the equation to check that you are correct.

2. Evaluate the eventually periodic continued fractions:

   (a) $[1; 2, 3, 1, 2, 3, 1, 2, 3, \ldots]$

   (b) $[4; 3, 2, 1, 1, 1, 1, \ldots]$

3. Suppose $x = [1; a, 1, a, 1, a, \ldots]$ where $a \in \mathbb{N}$. Find x and verify directly that it is irrational.

4. (Hard) Prove the $\Rightarrow$ direction of Theorem 1.14

1.6 Diophantine Approximations of Irrational Numbers

The question of finding good rational approximations to irrational numbers is very old. The approximation $\pi \approx \frac{22}{7}$ has been known for thousands of years: Example 1.11.2 shows that it arises naturally from the consideration of continued fractions. In this section we make two definitions of what it means to be a best rational approximation of an irrational number, and we see how these are related to the convergents of continued fractions.

Definition 1.15. Let $p, q \in \mathbb{N}$, and let x be a positive irrational number.

1. We say that $\frac{p}{q}$ is a *best approximation to x of the first kind* if,

$$\forall a, b \in \mathbb{N} \text{ such that } \frac{a}{b} \neq \frac{p}{q} \text{ and } b \leq q \text{ we have } \left| x - \frac{p}{q} \right| < \left| x - \frac{a}{b} \right|$$

   Of all fractions with denominators less than or equal to q, the fraction $\frac{p}{q}$ is closest to x.

2. We say that $\frac{p}{q}$ is a *best approximation to x of the second kind* if,

$$\forall a, b \in \mathbb{N} \text{ such that } \frac{a}{b} \neq \frac{p}{q} \text{ and } b \leq q \text{ we have } |qx - p| < |bx - a|$$

   The distance of qx to the nearest integer is smaller than that of all other bx when $b < q$.

The accuracy of a best approximation of the second kind is weighted against the size of its denominator. It should be clear that a best approximation of either type must be a fraction in lowest terms.

It is fairly easy to ask a computer to generate such approximations, at least for small denominators:

in the tables below,

- $\frac{p}{q}$ is a best approximation of the first kind $\iff$ $\left|\pi - \frac{p}{q}\right|$ is smaller than all entries above it.
- $\frac{p}{q}$ is a best approximation of the second kind $\iff$ the non-integer error in $q\pi$ is smaller than all entries above it.

Examples 1.16. We produce tables indexed by denominators q of the rational number $\frac{p}{q}$ closest to x and whether these are best approximations: everything is rounded to 5 d.p.

1. For $x = \sqrt{2} = 1.41421\ldots$

q	closest $\frac{p}{q}$	$\left\lvert\sqrt{2} - \frac{p}{q}\right\rvert$	first kind?	$q\sqrt{2}$	second kind?
1	$\frac{1}{1} = 1$	0.41421	✓	$1 + 0.41421$	✓
2	$\frac{3}{2} = 1.5$	0.08579	✓	$3 - 0.17157$	✓
3	$\frac{4}{3} = 1.33333$	0.08088	✓	$4 + 0.24264$	
4	$\frac{6}{4} = 1.5$	n/a			
5	$\frac{7}{5} = 1.2$	0.01421	✓	$7 + 0.07107$	✓
6	$\frac{8}{6} = 1.33333$	n/a			
7	$\frac{10}{7} = 1.42857$	0.01436		$10 - 0.10051$	
8	$\frac{11}{8} = 1.375$	0.03921		$11 + 0.31371$	
9	$\frac{13}{9} = 1.444\ldots$	0.03023		$13 - 0.27208$	

The following best approximation is of both kinds:

$$\frac{17}{12} = \sqrt{2} + 0.00245, \qquad 12\sqrt{2} = 17 - 0.02944$$

2. Consider $x = \pi = 3.14159\ldots$

q	closest $\frac{p}{q}$	$\left\lvert\pi - \frac{p}{q}\right\rvert$	first kind?	$q\pi$	second kind?
1	$\frac{3}{1}$	0.14159	✓	$3 + 0.14159$	✓
2	$\frac{6}{2} = 3$	n/a			
3	$\frac{9}{3} = 3$	n/a			
4	$\frac{13}{4} = 3.25$	0.10841	✓	$13 - 0.43363$	
5	$\frac{16}{5} = 3.2$	0.05841	✓	$16 - 0.29204$	
6	$\frac{19}{6} = 3.16667$	0.02507	✓	$19 - 0.15044$	
7	$\frac{22}{7} = 3.14286$	0.00126	✓	$22 - 0.00885$	✓
8	$\frac{25}{8} = 3.125$	0.01659		$25 + 0.13274$	
9	$\frac{28}{9} = 3.11111$	0.03048		$28 + 0.27433$	

$\frac{22}{7}$ is such a good approximation that it takes a while to find the next one of either type:

First kind: $\frac{179}{57} = \pi - 0.00124\ldots$

Second kind: $\frac{333}{106} = \pi - 0.0000832\ldots \qquad 106\pi - 333 = 0.00882\cdots$

You shouldn't be surprised to recall that $\frac{333}{106}$ is a convergent of π!

Indeed these examples should immediately suggest a couple of patterns...

Theorem 1.17. *Let x be irrational.*

1. *Best approximations of the second kind to x are also of the first kind.*
2. *The convergents of x are precisely the best approximations of the second kind.*[a]

[a] With one caveat: if $q_2 = 1$ (equivalently $c_2 = 1$) then $\frac{p_1}{q_1} = p_1 = c_1$ is not a best approximation of the second kind, since $\frac{p_2}{q_2} = p_2 = c_2 = c_1 + 1$ will be a closer approximating integer than c_1.

Proof. 1. Suppose $\frac{p}{q}$ is a best approximation of the second kind to x. Then $|qx - p| < |bx - a|$ for all $b \leq q$ where $\frac{a}{b} \neq \frac{p}{q}$. But then

$$\left| x - \frac{p}{q} \right| = \frac{1}{q} |qx - p| < \frac{1}{q} |bx - a| = \frac{b}{q} \left| x - \frac{a}{b} \right| \leq \left| x - \frac{a}{b} \right|$$

It follows that $\frac{p}{q}$ is a best approximation of the first kind.

2. Suppose $\frac{a}{b} \neq \frac{p_n}{q_n}$ with $1 \leq b \leq q_n$. By Exercise 2, $q_{n+1} \geq q_n$ with equality if and only if $n = 1$ and $c_2 = 1$, exactly the caveat in the theorem. We may therefore assume that $q_{n+1} > q_n \geq b$.

Following Lagrange, we consider solutions to a system of linear equations.

$$\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Since the determinant of the matrix is ± 1, there is a unique *integer* solution $\left(\begin{smallmatrix} y \\ z \end{smallmatrix} \right)$. We check cases:

(a) If $= z = 0$ we have a contradiction, for $b \neq 0$.
(b) If $y = 0$ and $z \neq 0$, then $b = q_{n+1} z \implies b \geq q_{n+1}$, a contradiction.
(c) If $z = 0$ and $y \neq 0$, then $yp_n = a$ and $yq_n = b$ forces $\frac{a}{b} = \frac{p_n}{q_n}$, another contradiction.
(d) If $y, z \neq 0$, then $0 < b = yq_n + zq_{n+1} \leq q_n$ forces y, z to have opposite signs. Since the convergents alternate either side of x, we see that $q_n x - p_n$ and $q_{n+1} x - p_{n+1}$ also have opposite signs. Therefore $y(q_n x - p_n)$ and $z(q_{n+1} x - p_{n+1})$ have the *same sign* and so

$$\begin{aligned} |bx - a| &= |(yq_n + zq_{n+1})x - (yp_n + zp_{n+1})| = |y(q_n x - p_n) + z(q_{n+1} x - p_{n+1})| \\ &= |y(q_n x - p_n)| + |z(q_{n+1} x - p_{n+1})| && \text{(terms have same sign)} \\ &> |y(q_n x - p_n)| \geq |q_n x - p_n| \end{aligned}$$

In the non-contradictory case, the conclusion reads $b \leq q_n \implies |bx - a| > |q_n x - p_n|$, whence $\frac{p_n}{q_n}$ is a best approximation of the second kind.

A small modification allows us to obtain the converse! Assume $\frac{a}{b}$ is *not* a convergent of x. Since (q_n) is increasing, we may assume that $q_n \leq b < q_{n+1}$ for some n. Considering the matrix equation, we see that all four cases of the analysis hold, as does the conclusion

$$|bx - a| > |q_n x - p_n|$$

Since $q_n \leq b$ it follows that $\frac{a}{b}$ is not a best approximation of the second kind. ∎

It can moreover be proved that the best approximations of the first kind are the convergents and some of their *intermediate fractions*; rational numbers of the form

$$\frac{p_{n+1}m + p_n}{q_{n+1}m + q_n} = [c_1; c_2, \ldots, c_{n+1}, m] \tag{$*$}$$

where m is an integer (roughly) between $\frac{1}{2}c_{n+2}$ and c_{n+2}. We won't pursue this, except to note that $\frac{179}{57} = \frac{22\cdot 8+3}{7\cdot 8+1}$ follows this pattern when considering π.

The primary application however is done. In the days before calculators, if we want a rational approximation of a number of desired accuracy, one needed only compute whichever convergent was necessary to achieve this. The same can even be done for rational numbers, provided you're a bit more flexible with non-strict inequalities.

Example 1.18. Suppose we want to find a good rational approximation to $\frac{317}{122}$ out of all fractions with denominator at most 10. First compute the continued fraction and its convergents:

$$\frac{317}{122} = [2; 1, 1, 2, 24]$$

n	0	1	2	3	4	5
c_n		2	1	1	2	24
p_n	1	2	3	5	13	317
q_n	0	1	1	2	5	122

Clearly $\frac{13}{5}$ is a good starting point: it is a best approximation of both kinds and is very accurate

$$\left|\frac{317}{122} - \frac{13}{5}\right| = \frac{1}{5\cdot 122} = \frac{1}{610} = 0.00163\ldots$$

as well as being much easier to work with due to its small denominator. In view of ($*$), we should probably also consider numbers of the form $\frac{13m+5}{5m+2}$ but the only one of these with denominator ≤ 10 is $\frac{18}{7}$ which isn't an approximation of the first kind.

Hurwitz's Theorem

In the next chapter we shall apply continued fractions and convergents to the famous *Pell equation* $x^2 - dy^2 = 1$ where $d \in \mathbb{N}$ is not a perfect square. To facilitate this we shall need the following results.

Theorem 1.19. *Suppose x is irrational.*

1. *There are infinitely many rational numbers $\frac{a}{b}$ satisfying*

$$\left|x - \frac{a}{b}\right| < \frac{1}{b^2}$$

2. *If $\frac{a}{b}$ is a rational number satisfing*

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$$

   then $\frac{a}{b}$ is a best approximation of the second kind to x (and thus a convergent of x). Moreover, at least one of each pair of successive convergents satisfies this inequality.

Proof. 1. This is Theorem 1.10, part 2: since $q_{n+1} \geq q_n$, *every* convergent of x satisfies the inequality.

2. Suppose $\frac{a}{b}$ satisfies the inequality but is not a best approximation of the second kind. Since (q_n) is increasing, there is some n such that

$$q_n \leq b < q_{n+1}$$

By Theorem 1.17, $\frac{p_n}{q_n}$ is closer to x (in the second sense) than $\frac{a}{b}$: applying the assumed inequality,

$$|q_n x - p_n| \leq |bx - a| < \frac{1}{2b} \implies \left| x - \frac{p_n}{q_n} \right| < \frac{1}{2bq_n}$$

But then

$$\begin{aligned} \frac{1}{bq_n} &\leq \frac{|aq_n - bp_n|}{bq_n} = \left| \frac{a}{b} - \frac{p_n}{q_n} \right| && (|aq_n - bp_n| \text{ is a positive integer since } \tfrac{a}{b} \neq \tfrac{p_n}{q_n}) \\ &\leq \left| x - \frac{a}{b} \right| + \left| x - \frac{p_n}{q_n} \right| && (\triangle\text{-inequality}) \\ &< \frac{1}{2b^2} + \frac{1}{2bq_n} \end{aligned}$$

But this is iff $q_n > b$, a contradiction. The claim about successive convergents is Exercise 7. ■

Note the distinction between the two parts of the theorem:

$$\left| x - \frac{a}{b} \right| < \frac{1}{2b^2} \implies \frac{a}{b} \text{ is a convergent of } x \implies \left| x - \frac{a}{b} \right| < \frac{1}{b^2}$$

A biconditional would be nice, but there are plenty of counterexamples: for instance as we'll see below, $\frac{11}{3}$ is a convergent of $\sqrt{13}$ but

$$\left| \sqrt{13} - \frac{11}{3} \right| \approx 0.0611\ldots \geq 0.0555\ldots = \frac{1}{2 \cdot 3^2}$$

The result can be improved as far as the following, though we won't prove it.

Theorem 1.20 (Hurwitz). *If x is irrational, then there are infinitely many rational numbers $\frac{a}{b}$ which satisfy*

$$\left| x - \frac{a}{b} \right| < \frac{1}{\sqrt{5}b^2}$$

All such are convergents of x, and at least one of every three successive convergents satisfies the inequality.

By thinking about the convergents of the golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$, one can see that $\sqrt{5}$ is the *largest* number which can be placed in the denominator without ruling out some examples.

Example 1.21. Here is a table of the first ten convergents of $\sqrt{13}$ with a check on the accuracy of $\left|\sqrt{13} - \frac{p_n}{q_n}\right|$ each time: note that the first convergent $\frac{p_1}{q_1} = 3$ is not a best approximation, so we should ignore it.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
p_n	3	4	7	11	18	119	137	256	393	649	4287	4936	9223
q_n	1	1	2	3	5	33	38	71	109	180	1189	1369	2558
$\left\lvert\sqrt{13} - \frac{p_n}{q_n}\right\rvert < \frac{1}{2q_n^2}$?		✓	✓		✓		✓	✓		✓		✓	✓

In fact all of the convergents checked also satisfy $\left|\sqrt{13} - \frac{p_n}{q_n}\right| < \frac{1}{\sqrt{5}q_n^2}$.

Exercises 1. Prove the claim that a best approximation of either type must be a fraction in lowest terms.
(*Hint: consider* $\frac{ak}{bk}$ *where* $\gcd(a, b) = 1$)

2. Suppose x is irrational and that a is an integer such that $|x - a| < \frac{1}{2}$: explain (*without using Theorem 1.19*) why a is a best approximation of the second kind to x.

3. Suppose that $\frac{a}{b} \neq \frac{c}{d}$ are both rational, and that $\left|x - \frac{a}{b}\right| = \left|x - \frac{c}{d}\right|$. Prove that $x \in \mathbb{Q}$.

   Repeat if $|bx - a| = |dx - c|$.

4. A Pell equation is an equation $x^2 - dy^2 = 1$ for *integers* x, y, where d is a positive integer that is not a perfect square. Explain why the equation is uninteresting if d is a perfect square.
(*If* $d = a^2$, *what are the integer solutions?*)

5. Prove that the following statements hold for every pair of positive integers (x, y):

   (i) $\left|x^2 - 2y^2\right| \geq 1$

   (ii) If $\left|x - y\sqrt{2}\right| < \frac{1}{y}$, then $x + y\sqrt{2} < 2y\sqrt{2} + \frac{1}{y}$.

   Now use (i) and (ii) to show that

$$\left|x - y\sqrt{2}\right| > \frac{1}{2y\sqrt{2} + \frac{1}{y}}$$

6. Suppose $x = [c_1; c_2, \ldots]$ is irrational, where $c_n \geq 2$ for all $n \geq 2$. Prove that

$$\frac{a}{b} \text{ is a convergent} \iff \left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$$

7. Prove the simplified version of Hurwitz's Theorem: at least one of every two successive convergents of x satisfies the inequality $\left|x - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2}$.
Hint: Recall that $\frac{p_n}{q_n}$ *and* $\frac{p_{n+1}}{q_{n+1}}$ *lie on either side of* x, *so*

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right|$$

8. Let $\varphi = \frac{1}{2}(1+\sqrt{5})$ be the *Golden Ratio.*

   (a) Compute the first few convergents $\frac{p_n}{q_n}$ of the continued fraction for φ. You should recognize the Fibonacci numbers: prove that this is really the correct pattern.

   (b) With reference to the intermediate fractions formula on page 16, argue that best approximations of the first and second kind are identical for φ.

   (c) Use Binet's formula

   $$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

   for the n^{th} Fibonacci number to prove that

   $$\left|\varphi - \frac{F_{n+1}}{F_n}\right| = \frac{1}{\sqrt{5}F_n^2}\left(1 - \frac{(-1)^n}{\varphi^{2n}}\right)$$

   from which exactly half the convergents satisfy Hurwitz's Theorem.

   (d) (Hard: for analysis aficionados only!) Prove that we cannot replace $\sqrt{5}$ with any larger number in Hurwitz's Theorem if we want it to apply to *all* irrationals. That is, if we want all irrationals x to have infinitely many rationals $\frac{a}{b}$ satisfying

   $$\left|x - \frac{a}{b}\right| < \frac{1}{kb^2}$$

   then we must have $k \le \sqrt{5}$. In this sense Hurwitz's Theorem is an optimal result.
   (*Hint: think about the definition of limit…*)

The above helps to explain why φ is one of the hardest numbers to approximate efficiently with rational numbers. If you think carefully about the Euclidean Algorithm you should see a relationship; for numbers of at most a given size, the Euclidean Algorithm will take the largest number of steps to compute the gcd of consecutive Fibonacci numbers.