Math 180B: Homework 3-1

Nothing to submit: just extra practice for the midterm

1. Find all the primitive roots modulo $54 = 2 \cdot 3^3$. Do this systematically, as we did in class rather than by guessing.

2. Solve the following linear recurrence relations using the characteristic equation method.
   
   (a) $a_n = 3a_{n-1} + 10a_{n-2}$, $a_1 = 1$, $a_2 = 3$
   (b) $b_n = 2b_{n-1} - 5b_{n-2}$, $b_1 = 1$, $b_2 = -3$
   (c) $c_n = 4c_{n-1} - c_{n-2} - 6c_{n-3}$, $c_1 = 0$, $c_2 = 0$, $c_3 = 1$

3. For each of the recurrences in the previous question, find the (eventual) periods modulo 2, modulo 3 and modulo 6. Check that $N(2 \cdot 3) = \text{lcm}(N(2), N(3))$ in each case.

4. (a) Solve the recurrence relations for the Pell numbers and the Pell-Lucas numbers $U_n$ and $V_n$ as defined in the notes.
   (b) Use part (a) to find an explicit expression for the $n$th convergent of $\sqrt{2}$ and an alternative proof that the convergents really do converge to $\sqrt{2}$.
   (c) For which primes $p$ will it be possible to find an explicit formula (of the type in part (a)) for $U_n$ and $V_n$ modulo $p$? Find an explicit formula modulo $p = 7$ (Think about quadratic residues...)
   (d) Compute the periods of $U_n$ and $V_n$ modulo 2, 3, 5, and 7. What do you expect the period modulo 210 to be? If you have the time, check it!

5. Let $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ define the Fibonacci sequence.
   (a) Prove that for all $n \geq k + 2$ we have $F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}$
   (b) Prove that $F_k \mid F_{2k}$ for all $k$.
   (c) More generally, prove that $k \mid n \implies F_k \mid F_n$.
   (d) Complete the statement and prove the following theorem:

   If $F_n$ is prime, then $n$ is...