1 Ancient Egypt

Summary

- Recorded civilization in Nile valley dates from c. 3150 BC.
- Conquered 322 BC by Alexander the Great (Macedonia/Greece) who left his general Ptolemy in charge. Cleopatra's death in 30 BC marked the end of the Ptolemaic dynasty; Egypt became a Roman province.
- Translation of ancient Egyptian writing was facilitated by the AD 1799 discovery of the Rosetta Stone, containing Greek, hieroglyphic and demotic (late cursive) Egyptian script.
- Primary mathematical sources: Rhind/Ahmes (A'h-mose) papyrus c. 1650 BC and the Moscow papyrus c. 2000 BC.[1] The Rhind papyrus contains around a hundred worked problem and two tables: unit fraction representations of $\frac{2}{n}$ for integers $n < 100$, and expressions for $\frac{3}{10}, \frac{4}{10}, \ldots, \frac{9}{10}$ in terms of unit fractions.

  The Moscow papyrus is shorter and more focused on geometric problems.
- Few other primary sources. Egyptians wrote on papyrus; as a plant-based early form of paper, which naturally decomposes. Other Egyptian mathematics was likely absorbed uncredited by the Greeks.
- Egyptian mathematics was practical & non-abstract. Worked problems on sums, linear equations, construction and land-measurement (tax-collection). No clear distinction between exact and approximate solutions.

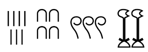
Part of the Rhind Papyrus

[1] Rhind was a Scottish egyptologist. Ahmes was the name of the scribe who wrote/copied the papyrus. The Moscow papyrus is named because it was sold to the Moscow Museum of Fine Art; its author is unknown.

Notation

Ancient Egypt had multiple systems of enumeration. These approaches changed over time, so we give only a loose overview.

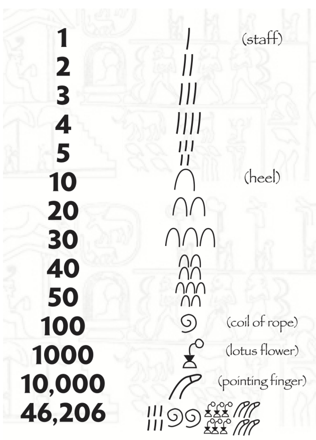
Hieroglyphic enumeration Dating from at least 5000 BC, a separate pictorial symbol was used for each power of 10. These were grouped and stacked to make larger numbers. Digits could be written in any order, left-to-right, right-to-left, or even vertically: e.g.

2347 = 

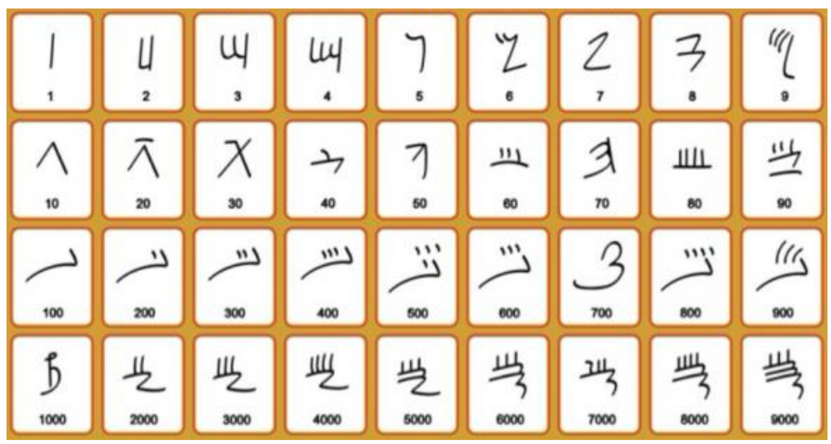
Hieroglyphs were cumbersome to use for calculations since large numbers needed many symbols, but were more appropriate for carving into stone.

Hieratic enumeration A cursive system dating from c. 2000 BC, as seen on the Rhind papyrus. Symbols for 1–9, 10–90, 100–900 etc., were mapped onto the hieratic alphabet, an approach copied later by the Greeks with their own alphabet.

As suggested in the table below, 23 would be written (approximately) |||$\overline{\wedge}$.



Hieroglyphic Enumeration



Hieratic Enumeration

There was no numeral or symbol for zero, though the hieroglyph *nfr* 𓄤, meaning beautiful or perfect, was used to indicate balanced books in accounting or the base level of a building.

Egyptian Fractions

The Egyptians had several notations for fractions. One of the oldest versions (c. 2500 BC) used parts of the *Eye of Horus* 𓂀 to indicate repeated halving: for instance

$$\triangleleft = \frac{1}{2}, \qquad \circ = \frac{1}{4}, \qquad \frown = \frac{1}{8}, \qquad \triangleright = \frac{1}{16}, \qquad \text{etc.}$$

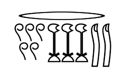
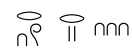
This was very cumbersome, and impractical for computation.

Later Egyptian notation expanded this idea: while we are used to both decimal and fractional notation, e.g. $0.95 = \frac{19}{20}$, ancient Egyptians would instead have viewed this quantity as a sum of *reciprocals of integers*:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5}$$

Quantities represented thus are still called *Egyptian fractions*, and are actively researched.
In hieratic notation, a dot denoted the reciprocal: e.g., $\dot{\wedge}$ is $\frac{1}{10}$. In hieroglyphs, an oval ⬭ was used instead. To get a feel for this approach, we indicate reciprocals with a bar.

As with integers, combinations of fractions could be written in any order/direction. Examples in both hieroglyphic and a suitably modernised notation are shown in the table.
The fraction $\frac{1}{2}$ received the special symbol . Late in Egyptian times, two others did also:  for $\frac{2}{3}$ and  for $\frac{3}{4}$.

Quantity	Hieroglyph	Modern
$\frac{1}{3}$		$\overline{3}$
$\frac{1}{45}$		$\overline{45}$
$\frac{1}{23500}$		$\overline{23500}$
$30 + \frac{1}{2} + \frac{1}{110}$		$\overline{110}\ \overline{2}\ 30$

The Rhind papyrus contains a table showing how to express $\frac{2}{n}$ as Egyptian fractions for all odd integers $n < 100$. The first column denotes n, and the remaining columns the Egyptian fraction representation. For instance, the first row states

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$$

5	3	15	
7	4	28	
9	6	18	
11	6	66	
13	8	52	104
15	10	30	

The table speeds the computation of harder fractions. For instance,

$$\frac{5}{13} = 2 \cdot \frac{2}{13} + \frac{1}{13} = 2\left(\frac{1}{8} + \frac{1}{52} + \frac{1}{104}\right) + \frac{1}{13} = \frac{1}{4} + \frac{1}{13} + \frac{1}{26} + \frac{1}{52}$$

There are several approaches to finding Egyptian fraction representations; moreover, any positive rational number may be written in such a form. If the denominator is odd, we could choose

$$\frac{2}{mn} = \frac{1}{mr} + \frac{1}{nr} \quad \text{where} \quad r = \frac{m+n}{2}$$

Many, but not all, of the lines in the Rhind table follow this formula.

- The line for $\frac{2}{5}$ has $m = 1$, $n = 5$ and $r = 3$; this is unique up to reordering.
- This formula permits multiple representations. For instance

  $$\frac{2}{9} = \begin{cases} \frac{1}{9} + \frac{1}{9} & (m,n,r) = (3,3,3) \\ \frac{1}{5} + \frac{1}{45} & (m,n,r) = (1,9,5) \end{cases}$$

  The first is essentially useless, whereas the second isn't the Rhind expression!

- In the Rhind table, $\frac{2}{13}$ is written as a sum of three reciprocals instead of two.

Calculations & Example Problems

Egyptian notation is computationally inefficient. Hieroglyphs were slow to write and cursive notation required many symbols. Regardless, we have some evidence of practical calculation methods and many worked problems.

Addition/Subtraction

In practice this would likely have been done with carrying and borrowing, though details are scant. A special symbol was used to denote both addition and subtraction, with its meaning changing depending on the direction the text was read.

Carrying and borrowing would have been very simple with hieroglyphs: gather all symbols in a sum and replace ten of one symbol by the next. For subtraction, one might need to convert a larger symbol into ten of a smaller one.

Multiplication

This relied on a base-two algorithm. To compute the product ab, we create a table:

1. In the first row write 1, b.
2. In successive rows, double the previous row until the first term is about to exceed a.
3. Determine the powers of 2 that sum to a.
4. Sum the corresponding multiples of b.

For example, to compute $13 \cdot 15$, we construct a table where the checked rows are summed.

$$\begin{array}{rrc} 1 & 15 & \checkmark \\ 2 & 30 & \\ 4 & 60 & \checkmark \\ 8 & 120 & \checkmark \end{array} \quad \Longrightarrow \quad 13 \cdot 15 = (1+4+8) \cdot 15 = 15 + 60 + 120 = 195$$

Note how $13 = 1 + 4 + 8 = 2^0 + 2^2 + 2^3$ is essentially the binary representation. We stopped at the fourth row since another doubling would have resulted in the first term (16) exceeding 13. We could instead have reversed the roles of the factors:

$$\begin{array}{rrc} 1 & 13 & \checkmark \\ 2 & 26 & \checkmark \\ 4 & 52 & \checkmark \\ 8 & 104 & \checkmark \end{array} \quad \Longrightarrow \quad 15 \cdot 13 = (1+2+4+8) \cdot 13 = 13 + 26 + 52 + 104 = 195$$

All you need is addition and the ability to multiply by 2. In the abstract, this is the distributive law combined with the binary representation of a:

$$a = \sum_{j=0}^{n} a_j 2^j \implies ab = \sum_{j=0}^{n} a_j 2^j b \qquad (\text{each } a_j \in \{0, 1\})$$

Division

This also relies on doubling and halving, though the answer is non-unique. To compute $\frac{a}{b}$, consider the problem $a = bx$ and apply a variant of the multiplication algorithm to find multiples of b summing to a. Here are two examples.

1. To compute $260 \div 13$, we repeatedly double 13 until terms in the right column sum to 260.

1	13	
2	26	
4	52	✓
8	104	
16	208	✓

   Since $260 = 208 + 52$, we conclude that $260 \div 13 = 16 + 4 = 20$.

2. To find $5 \div 13$, divide by 2 with the intent of making terms in the right column sum to 5.

1	13	
$\overline{2}$	$6\,\overline{2}$	
$\overline{4}$	$3\,\overline{4}$	✓
$\overline{8}$	$1\,\overline{2}\,\overline{8}$	✓

   Since $\left(3\,\overline{4}\right) + \left(1\,\overline{2}\,\overline{8}\right) = 4\,\overline{2}\,\overline{4}\,\overline{8}$ is $\overline{8}$ $(\frac{1}{8})$ short of what we want, we continue the table differently. Divide the first row by 13 and continue halving until we obtain the desired $\overline{8}$ in the right column.

$\overline{13}$	1	
$\overline{26}$	$\overline{2}$	
$\overline{52}$	$\overline{4}$	
$\overline{104}$	$\overline{8}$	✓

$$\implies 5 \div 13 = \overline{4}\,\overline{8}\,\overline{104} = \frac{1}{4} + \frac{1}{8} + \frac{1}{104}$$

   In practice, a scribe likely used a table such as that in the Rhind papyrus to obtain the same answer (see page 3). Note also that the method is non-unique: we could instead have checked the third, fifth, sixth and seventh rows of the above table to obtain

$$5 = (3\,\overline{4}) + 1 + \overline{2} + \overline{4} \implies 5 \div 13 = \overline{4}\,\overline{13}\,\overline{26}\,\overline{52}$$

Practical application: Loaf-splitting A typical Egyptian problem might involve determining how to split 5 loaves among 13 people. The previous calculation says that we could give each person $\frac{1}{4} + \frac{1}{8} + \frac{1}{104}$ of a loaf. This might seem complicated, but it has some advantages over a typical modern solution (cut each loaf into 13 equal pieces and give each person five):

- The Egyptian approach of repeatedly cutting in half is easy to do with reasonable accuracy and fairness (at least for the one-fourth and one-eighth pieces). Cutting a loaf into 13^{th} parts is difficult! The remaining 104^{th} parts of a loaf would probably be ignored as crumbs.
- Even with full accuracy, the Egyptian approach requires 34 cuts, reduced to 22 if the 104^{th} parts are ignored. The modern approach requires 60 cuts.

Linear equations

Solutions were based on the method of *false position*: guess an approximate solution and modify until it works. Here is problem 24 of the Rhind papyrus.

> A heap plus a seventh of a heap is nineteen. What is the heap?

In modern algebra, we wish to solve $x + \frac{1}{7}x = 19$ where x is the 'heap.'

1. Guess intelligently: $x = 14$ is easy to divide by 7 and produces $x + \frac{1}{7}x = 16$.
2. Modify our guess: We want 19, so we multiply our guess (14) by $\frac{19}{16} = 1\,\overline{8}\,\overline{16}$ to obtain the correct answer

$$x = 14\big(1\,\overline{8}\,\overline{16}\big) = 2\,\overline{4}\,\overline{8} + 4\,\overline{2}\,\overline{4} + 9\,\overline{2} = 16\,\overline{2}\,\overline{8}$$

1	$1\,\overline{8}\,\overline{16}$	
2	$2\,\overline{4}\,\overline{8}$	✓
4	$4\,\overline{2}\,\overline{4}$	✓
8	$9\,\overline{2}$	✓

Compare this to the 'modern' method:

$$x + \frac{1}{7}x = 19 \implies \frac{8}{7}x = 19 \implies x = \frac{7 \cdot 19}{8} = \frac{133}{8} = \frac{128+5}{8} = 16\frac{5}{8}$$

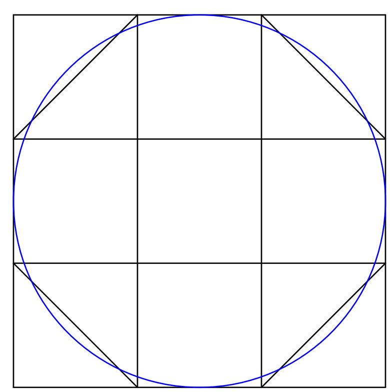
Are the any benefits to the Egyptian approach?

Geometry

Problem 48 of the Rhind papyrus uses an octagon to approximate the area of a circle.

A square of side 9 is drawn, where each side is split into thirds and the four corner squares are cut in half. The area of the octagon is then

$$81 - 18 = 63$$

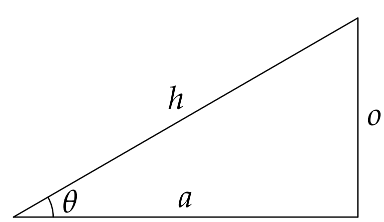


Since the area of the circle is $\frac{81\pi}{4}$, this amounts to the approximation $\pi \approx \frac{28}{9} = 3.11111\ldots$

Problem 50 compares the same circle to a square of side 8. This time $\frac{81\pi}{4} \approx 64$ corresponds to $\pi \approx \frac{256}{81} = 3.16049\ldots$

No explanation is given as to what inspired these methods, nor whether the scribes understood that these were only approximations.

Other problems computed or approximated areas and volumes of triangles, quadrilaterals, boxes, cylinders and truncated pyramids, all via worked examples without general formulæ. The Egyptians even had a notion of cotangent which they called the *seked*, useful for describing and calculating slopes.



$$\text{seked} = \frac{a}{o} = \cot\theta$$

Exercises Don't memorise hieroglyphic or hieratic notation! Use modern bar-notation to denote Egyptian fractions and illustrate algorithms.

1. Use Egyptian techniques to compute:
   (a) Multiply 7 by 12.
   (b) Multiply 34 by 18.
   (c) Divide 93 by 5.
   (d) Divide 102 by 7.

2. Use Egyptian techniques to multiply $\overline{28}$ by $1 + \overline{2} + \overline{4}$ (problem 9 of the Rhind papyrus).

3. Part of the Rhind papyrus table for $\frac{2}{n}$ reads

   $$2 \div 11 = \overline{6} + \overline{66}$$
   $$2 \div 13 = \overline{8} + \overline{52} + \overline{104}$$
   $$2 \div 23 = \overline{12} + \overline{276}$$

1	13
$\overline{2}$	$6\,\overline{2}$
$\overline{4}$	$3\,\overline{4}$
$\overline{8}$	$1\,\overline{2}\,\overline{8}$
$\overline{52}$	$\overline{4}$
$\overline{104}$	$\overline{8}$
$\overline{8}\,\overline{52}\,\overline{104}$	$1\,\overline{2}\,\overline{4}\,\overline{8}\,\overline{8}$
	2

   The calculation of $2 \div 13$ is shown in a modern rendering; the terminal '2' is a verification of the calculation, that the last entry in the right-hand column really is 2.

   Perform similar calculations for $2 \div 11$ and $2 \div 23$.

   (*If you want the exact results from the papyrus you'll need* $\frac{2}{3}$*: denote this by* $\overline{\overline{3}}$)

4. Draw a picture of 5 loaves to help describe how the Egyptians might have divided them between seven people.

5. Use the method of false position to solve problem 28 of the Rhind papyrus:

   > A quantity and its 2/3 are added together; from this sum 1/3 of the sum is subtracted and 10 remains. What is the quantity?

6. Calculate a quantity such that if it is taken two times along with the quantity itself, the sum comes to 9 (problem 25 of the Moscow papyrus).

7. Use the method of false position to solve the algebraic equation $2x + \frac{1}{4}x = 15$.

   (*For more of a challenge, try to write any required calculations in Egyptian style*)

8. (a) Find all the ways in which $\frac{2}{13} = \frac{1}{a} + \frac{1}{b}$ can be written as a sum of reciprocals of positive integers a and b.
   (b) Repeat your calculation for $\frac{2}{p}$ whenever p is an odd prime.