2 Babylon/Mesopotamia

Babylon was an ancient city located near modern-day Baghdad, Iraq. The term also serves as a shorthand for the many Mesopotamian[2] empires/civilizations dating back at least to 3000 BC: Sumeria, Akkadia, Babylonia, etc.

The Babylonians used *cuneiform* (wedge-shaped) script, often via indentations on clay tablets. In contrast to Egyptian papyri, hundreds of thousands of tablets have survived to modern times. A significant proportion date from the time of Hammurabi (c. 1800 BC) or the Seleucid dynasty (c. 300 BC) which ruled after the conquests of Alexander the Great.

Mathematical tablets are of two main types: tables of values (multiplication, reciprocals, measures) and worked problems.

Mesopotamia and the Fertile Crescent

Sexagesimal Positional Enumeration

Our modern decimal system is *base 10 positional*. Both concepts are easy to understand with reference, say, to the number 3835.

Positional The single symbol 3 represents both 3000 and 30: the **meaning of a symbol depends on its position**.

Base 10 The position of a symbol denotes the **power of 10** by which it should be multiplied. Thus

$$3835 = 3 \cdot 10^3 + 8 \cdot 10^2 + 3 \cdot 10^1 + 5 \cdot 10^0$$

[2]'Between two rivers,' namely the Tigris and Euphrates. As indicated on the map, these rivers formed the backbone of the *fertile crescent*, a region of early civilization, farming, crop and animal domestication.

Positional enumeration makes for efficient calculations and easy representations vastly different magnitudes. By contrast, Egyptian notation is non-position: for instance ∩ means 10 wherever it is placed. Contrast with the difficulty of performing calculations using (non-positional) Egyptian hieroglyphic notation.

A key Babylonian contribution to mathematical history is the creation of one of the first positional systems of enumeration, dating at least to 2000 BC. Rather than our ten symbols 0–9, Babylonians used only two: roughly 𒐕 for 1 and 𒌋 for 10, likely made by the same stylus. Any number up to 59 could be written with combinations of these symbols, e.g.,

53 = 𒐐𒐗

The picture shows a typical cuneiform representation. Other numbers were represented base 60. Just as how, in decimal notation, the symbol 3 might mean 3000 or $\frac{3}{1000}$ depending on its position, the Babylonian 𒐗 could mean 3 times any power of 60, including fractions: e.g.

$$3, \quad 180 = 3 \cdot 60, \quad 10800 = 3 \cdot 60^2, \quad \frac{1}{20} = 3 \cdot 60^{-1}, \quad \frac{1}{1200} = 3 \cdot 60^{-2}$$

Such expressions were combined to describe all manner of numbers: for instance, the sexagesimal decomposition of 3835 is

$$3835 = 1 \cdot 60^2 + 3 \cdot 60^1 + 55 \cdot 60^0$$

which the Babylonians would have written

𒐕 𒐗 𒐐𒐙 (∗)

To make things easier to read, while retaining some of the feel of Babylonian notation, we'll use standard Hindu-Arabic numerals with commas to separate sexagesimal places and, if necessary, a semicolon to denote the sexagesimal point: thus

$$3835 = 1, 3, 55;$$

Fractions were expressed in the same way we use decimals:

$$23, 12; 1, 15 = 23 \cdot 60 + 12 + \frac{1}{60} + \frac{15}{60^2} = 1392\frac{1}{48}$$

There was no symbol for zero (as a placeholder) until very late in Babylonian history, nor any sexagesimal point, so determining position on ancient tablets can be difficult. Rather than 3835, (∗) might instead have represented

$$60 + 3 + \frac{55}{60} = 63\frac{11}{12} \quad \text{or} \quad 60^3 + 3 \cdot 60^2 + 55 \cdot 60 = 230100$$

Why base-60?

There are many theories but we cannot be certain. Here are some ideas.

- The Babylonians possibly combined two systems (base 10 and base 12) inherited from older cultures.
- Since 60 has many proper divisors (1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30), more numbers have exact representation than with decimal arithmetic: for instance $\frac{1}{3} = 0;20$, as a terminating sexagesimal, is much simpler than the infinite decimal 0.33333 ....
- As prolific astronomers and astrologers, the Babylonians might have chosen 60 as a divisor of 360, approximately the number of days in a year. Our modern usage of *degrees-minutes-seconds* for angle, *hours-minutes-seconds* for time has Babylonian origins.[3]
- Babylonian units of measure often used factors of 60 for magnitude similarly to how modern science uses 1000 (e.g., joules → kilojoules → megajoules).

This sort of historical question is rarely answerable in a satisfying way. Almost certainly there was no single moment where the Babylonians decided to use base 60. Like most cultural issues, it likely happened slowly and organically, without fanfare.

Basic Sexagesimal Calculations

Addition and subtraction would have been as natural to the Babylonians as decimal calculations are to us. For instance, we might write

$$\begin{array}{rr} & 21,49 \\ + & \overset{}{\underset{1}{3}},37 \\ \hline & 25,26 \end{array} \qquad \text{(in decimals } 1309 + 217 = 1526\text{)}$$

Note how we carry 60 just like we are used to doing with 10 in decimal arithmetic: $49+37 = 1,26$.

Multiplication is significantly harder. To mimic our familiar long-multiplication process would require memorizing up to the 59 times table! For small factors this might have been fine. For larger factors there is evidence of the Babylonians using two representations of a product in terms of squares

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2] = \frac{1}{4}[(x+y)^2 - (x-y)^2]$$

The computation of large products was aided by tablets listing squares. For instance,

$$\begin{aligned} 31 \times 22 &= \frac{1}{4}[53^2 - 9^2] = \frac{1}{4}[46,49 - 1,21] = \frac{1}{4}[45,28] = \frac{1}{4}[44,28 + 1,0] \\ &= 11,7 + 15 = 11,22 \qquad (= 682) \end{aligned}$$

[3]Much of western astrology also has Babylonian roots. For instance, the word Taurus (bull) comes to us from ancient Greek, but the association of the constellation with a bull is Babylonian ("The Great Bull of Heaven").

Fractions & Division

As we've already seen, the Babylonians represented non-integers using sexagesimals. Division by an integer n was done via multiplication by its reciprocal $\frac{1}{n}$:

$$m \div n = m \times \frac{1}{n}$$

Pre-computed tables of reciprocals aided this process. For example

$$\frac{1}{18} = 0;3,20 \implies 23 \div 18 = 23 \times 0;3,20 = 1;9 + 0;7,40 = 1;16,40$$

This works nicely provided n has no prime divisors other than 2, 3 or 5, since any such $\frac{1}{n}$ will be an exact terminating sexagesimal.[4] Approximations were used for other reciprocals; a scribe would choose a nearby exact fraction and state that the answer was approximate: e.g.

$$\frac{11}{29} \approx \frac{11}{30} = 11(0;2) = 0;22$$

For more accuracy, a larger denominator was chosen. For instance, if a scribe wanted to divide by 11, they might observe that $11 \cdot 13 = 143 \approx 144$, from which[5]

$$\frac{1}{144} = 0;0,25 \implies \frac{1}{11} \approx \frac{13}{144} = 0;5,25 \qquad (\dagger)$$

Scribes were explicit in acknowledging the approximation: "11 does not divide." Remember that a single digit in the second sexagesimal place means only $\frac{1}{3600}$, so even the most demanding application doesn't require many terms: (†) is roughly a 0.7% underestimate!

Another table listed all the ways an integer < 10 could be multiplied exactly to get 10.

1	10	5	2
2	5	6	1 40
3	3 20	8	1 15
4	2 30	9	1 6 40

We omit the commas (they didn't exist anyway). Moreover 7 is missing since $\frac{1}{7}$ (and thus $\frac{10}{7}$) is not an exact sexagesimal. The table could be applied in various ways: for instance,

$$\frac{10}{6} = 1;40 \quad \text{and} \quad \frac{600}{9} = 1,6;40$$

In the second case, note that $600 = 10 \cdot 60$ would be written the same as 10, so this merely amounts to moving the sexagesimal point in the final table entry $\frac{10}{9} = 1;6,40$.

[4] Analogous to the fact that $\frac{1}{n}$ has a terminating decimal if and only if n has no prime divisors other than 2 or 5.

[5] As a rational number, $\frac{1}{11} = 0.09090909\ldots = 0;5,27,16,21,49,5,27,16,21,49,\ldots$ has a periodic sexagesimal expansion. This can be computed easily using a calculator:

$$\frac{60}{11} = 5 + \frac{5}{11}, \quad \frac{5 \cdot 60}{11} = 27 + \frac{3}{11}, \quad \frac{3 \cdot 60}{11} = 16 + \frac{4}{11}, \quad \frac{4 \cdot 60}{11} = 21 + \frac{9}{11}, \ldots$$

Linear Systems of Equations

Like the Egyptians, the Babylonians also applied the method of false position (guess and modify). While the Eygptians applied the method to linear equations in a single variable, the Babylonians dealt with multiple.[6] For instance, here is a suitably modernized Babylonian approach to solving the linear system

$$\begin{cases} 3x + 2y = 11 \\ 2x + y = 7 \end{cases}$$

1. Choose one equation, say the second, and set $\hat{x} = \hat{y}$. Solve the resulting linear equation $3\hat{x} = 7$ (using false position or otherwise) to obtain $\hat{x} = \hat{y} = \frac{7}{3} = 2;20$.
2. Since $(d, -2d)$ is the general solution to the homogeneous equation $2x + y = 0$, all solutions to the second equation necessarily have the form

   $$x = \hat{x} + d, \qquad y = \hat{y} - 2d$$

   where d is, as yet, unknown. Substitute these expressions into the first equation to obtain a linear equation for d (if necessary, this could also be solved by false position):

   $$11 = 3\left(\frac{7}{3} + d\right) + 2\left(\frac{7}{3} - 2d\right) = 11 + \frac{2}{3} - d \implies d = \frac{2}{3}$$

3. Finally compute the answer

   $$x = \hat{x} + d = \frac{7}{3} + \frac{2}{3} = 3, \qquad y = \hat{y} - 2d = \frac{7}{3} - \frac{4}{3} = 1$$

Step 2 is essentially the nullspace method from modern linear algebra: all solutions to the matrix equation $(2\ 1)\begin{pmatrix} x \\ y \end{pmatrix} = 7$ have the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathbf{n} = \begin{pmatrix} 7/3 \\ 7/3 \end{pmatrix} + \begin{pmatrix} d \\ -2d \end{pmatrix}$$

where $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \end{pmatrix}$ is a particular solution and $\mathbf{n} = \begin{pmatrix} d \\ -2d \end{pmatrix}$ is a general element of the nullspace of the (row) matrix $(2\ 1)$.

[6]Babylonian mathematics certainly *appears* significantly more advanced than that of ancient Egypt. Remember, however, that the supply of primary sources is vastly different (half a million tablets versus a few papyri). While it seems likely the Babylonians were indeed more advanced, we cannot be certain that problems such as these were not also within the grasp of Egyptian scholars.

Pythagorean Triples

The Plimpton 322 tablet (kept at Yale) lists a large number of Pythagorean triples (albeit with some mistakes). Due to its strange encoding, it took scholars a long time to realize what they had. For instance, line 15 describes the Pythagorean triple $53^2 = 45^2 + 28^2$ as follows:

- The first entry $1;23,13,46,40$ is the exact sexagesimal for $\left(\frac{53}{45}\right)^2$.
- The second entry is 28.
- The third entry is 53.
- The last two entries indicate line number 15.

The Plimpton 322 tablet

The first three entries in each row are $\left((\frac{c}{a})^2, b, c\right)$ where $c^2 = a^2 + b^2$. Since the table is broken on the left side it is possible that one or more missing columns explicitly mentioned a.

It is not known how the table was completed, though the first column exhibits a descending pattern that provides clues to its construction. One theory is that a scribe found rational solutions to the equation $v^2 = 1 + u^2$ (equivalently $(v+u)(v-u) = 1$) by starting with a choice of $v+u$ and using a table of reciprocals to calculate $v-u$.

We revisit the line 15 example. If $v + u = \frac{9}{5} = 1;48$, then

$$v - u = \frac{1}{v+u} = \frac{5}{9} = 0;33,20$$

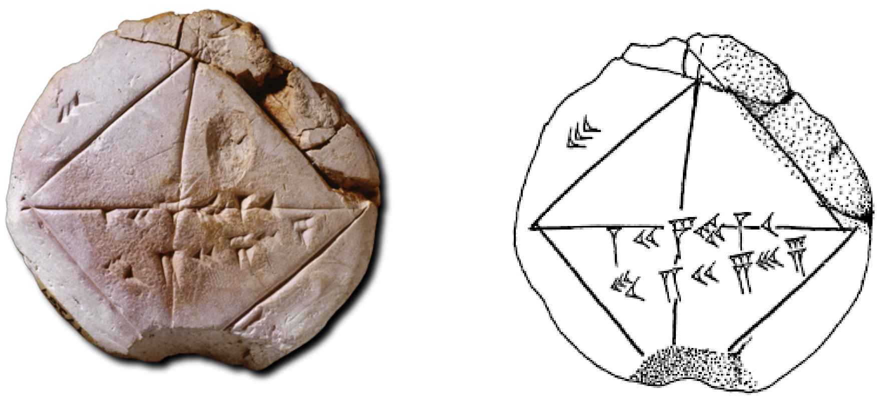
We therefore have a linear system of equations in u, v, which we know the Babylonians could solve:

$$\begin{cases} v + u = \frac{9}{5} \\ v - u = \frac{5}{9} \end{cases} \implies v = \frac{53}{45}\ (= 1;10,40), \quad u = \frac{28}{45}\ (= 0;37,20)$$

We investigate this further in Exercise 8. The Plimpton tablet has been the source of enormous scholarship; look it up!

Square-root Approximations & Quadratic Equations

The *Yale Babylonian Collection* contains over 45,000 objects, including YBC 7289, perhaps the most famous known mathematical tablet. It describes an incredibly accurate approximation to $\sqrt{2}$.



YBC 7289 and an enhanced representation of its numerals

The YBC 7289 tablet depicts a square of side 30 and labels the diagonal in two ways:

- $1;24,51,10$ is an approximation to $\sqrt{2}$. The accuracy of this is incredible, being an underestimate by roughly 1 part in 2.5 million!
- $42;25,35$ is the corresponding approximation to the diagonal when the side is 30.

The Babylonians more often used the simpler approximation $1;25 = 1.41666\ldots$.

Given the impractical accuracy of YBC 7289, it is reasonable to ask how it was found. No-one knows for certain, but two methods are theorized since both were used to solve other problems. It should be stressed that no Babylonian *proofs* of these approaches are known.

1. *Square root approximation* The expression

   $$\sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}$$

   is essentially the linear approximation familiar from elementary calculus (Exercise 7). If a is a rational number whose square is close to 2, then the error will be small. For instance:

   - $\sqrt{2} = \sqrt{1+1} \approx 1 + \frac{1}{2} = 1;30$ $\quad (a = 1)$
   - $\sqrt{2} = \sqrt{\left(\frac{4}{3}\right)^2 + \frac{2}{9}} \approx \frac{4}{3} + \frac{2/9}{8/3} = \frac{4}{3} + \frac{1}{12} = \frac{17}{12} = 1;25$ $\quad (a = \frac{4}{3} = 1.3333\ldots)$
   - $\sqrt{2} = \sqrt{\left(\frac{7}{5}\right)^2 + \frac{1}{25}} \approx \frac{7}{5} + \frac{1/25}{14/5} = \frac{99}{70} = 1;24,51,25,42,51,25,42,\ldots$ $\quad (a = \frac{7}{5} = 1.4)$

   The last expression, a period-three sexagesimal, is very close to the value on YBC 7289.

2. *Method of the Mean* It may be verified (Exercise 12) that any sequence defined by the recurrence relation

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

converges to $\sqrt{2}$. We apply this with initial term $a_1 = 1$.

$$a_1 = 1, \qquad a_2 = \frac{3}{2} = 1;30, \qquad a_3 = \frac{17}{12} = 1;25$$
$$a_4 = \frac{577}{408} = 1 + \frac{169}{408} = 1;24,51,10,35,17,\ldots$$
$$a_5 = \frac{665857}{470832} = 1;24,51,10,7,46\ldots$$

It seems incredible that any ancient culture would have found it useful to obtain such accuracy!

The same approach can be used to approximate other roots. For example, to approximate $\sqrt{11}$ we use via $a_{n+1} = \frac{1}{2}(a_n + \frac{11}{a_n})$ and $a_1 = 3$:

$$a_2 = \frac{10}{3} = 3;20, \qquad a_3 = \frac{199}{60} = 3;19,$$
$$a_4 = \frac{79201}{23880} = 3;18,59,50,57,17,\ldots$$

If you've studied number theory, you might recognize some of these values, say as the "best-approximations" arising from the study of continued fractions.

Quadratic Equations

The above methods were applied to solve general quadratic equations. Here is a typical question:

> I added twice the side to the square; the result is 2, 51, 40. What is the side?

In modern language, we want the solution to the quadratic equation

$$x^2 + 2x = 2 \cdot 60^2 + 51 \cdot 60 + 40 = 10300$$

Problems such as these were solved using templates, typically as worked examples. The above requires the template for $x(x+p) = q$ where $p, q > 0$. Since the Babylonians did not recognize negative numbers, the other types of quadratic equation ($x^2 = px + q$, etc.) had different templates.

To make things a little easier to follow, we describe the Babylonian template for $x(x+p) = q$ while simultaneously applying it to the simpler equation $x^2 + 4x = 2$:
Introduce a new variable $y = x + p$ $(y = x + 4)$ and decouple the equation:

$$\begin{cases} xy = q \\ y - x = p \end{cases} \qquad \begin{cases} xy = 2 \\ y - x = 4 \end{cases}$$

Solve for $x + y$ by completing the square:

$$\begin{aligned} 4xy + (y-x)^2 &= p^2 + 4q \\ (y+x)^2 &= p^2 + 4q \\ x + y &= \sqrt{p^2 + 4q} \end{aligned} \qquad \begin{aligned} 4xy + (y-x)^2 &= 4^2 + 4 \cdot 2 \\ (y+x)^2 &= 24 \\ x + y &= \sqrt{24} \approx 4;54 \end{aligned}$$

The square-root would have been approximated using one of the earlier algorithms, e.g.

$$\sqrt{24} = \sqrt{5^2 - 1} \approx 5 - \frac{1}{10} = 4;54$$

Since $x + y$ and $x - y$ are now known, we have an easily solved linear system:

$$x = \frac{\sqrt{p^2 + 4q} - p}{2} \qquad x = \frac{\sqrt{24} - 4}{2} \approx \frac{1}{2}(0;54) = 0;27$$

The method of completing the square and the quadratic formula are at least 4000 years old!

While we've written this abstractly, Babylonian scribes would typically have copied and edited a pre-existing example of the same type. There were no abstract formulæ and everything was done without the benefit modern notation. There was moreover often no written commentary to explain the method; historians had to decipher the approach from a single column of numbers!

Note also that the template only found the positive solution to the quadratic equation; the Babylonians had no notion of negative numbers. Amazingly, they were able to address certain *cubic* equations in a similar fashion.

Geometry

The Babylonians also considered many geometric problems. For instance:

- Approximations to areas of circles using both $\pi \approx 3$ and $\pi \approx 3\frac{1}{8}$.
- Calculations (correct and erroneous) for the volume of a frustrum (truncated pyramid).
- The altitude of an isosceles triangle bisects its base.
- The angle in semicircle is a right-angle (Thales' Theorem).

None of these statements or techniques were presented as theorems in a modern sense; we merely have computations and applications making use of such facts. We simply do not know the depth of Babylonian understanding of such concepts.

Exercises Remember that there is no single correct way to do Babylonian calculations. Play with the ideas and use modern notation to get a feel for things without torturing yourself.

1. Convert the sexagesimal values $0;22,30$, $0;8,6$, $0;4,10$ and $0;5,33,20$ into ordinary (modern) fractions in lowest terms.

2. Compute the products $25 \times 1,4$ and $18 \times 1,21$.

   (*Either compute directly* (*long multiplication*) *or use the difference of squares method on page 10*)

3. Use reciprocal multiplication to divide 50 by 18 and to divide 1, 21 by 32.

4. Use the Babylonian approach to solve the linear system

$$\begin{cases} 3x + 5y = 19 \\ 2x + 3y = 12 \end{cases}$$

5. (a) Convert the approximation $\sqrt{2} \approx 1;24,51,10$ to a decimal and verify the accuracy of the approximation on page 14.

   (b) Multiply by 30 to check that the length of the diagonal is as claimed.

6. (a) Use the square-root approximation page 14 with $a = \frac{8}{3}$ to approximate to $\sqrt{7}$.

   (b) Taking $a_1 = 3$, apply the method of the mean to find the approximation a_3 to $\sqrt{7}$.

7. Use elementary calculus to compute the linear approximation to the function

$$f(x) = \sqrt{a^2 + x}$$

   at $x = b$, and hence verify the square-root approximation (page 14).

8. Recall that $v^2 = 1 + u^2$ in the construction of the Plimpton tablet.

   (a) If $v + u = \alpha$, show that $u = \frac{1}{2}(\alpha - \alpha^{-1})$ and $v = \frac{1}{2}(\alpha + \alpha^{-1})$.

   (b) Suppose $v + u = 1;30 = \frac{3}{2}$. Find u, v and the corresponding Pythagorean triple.

   (c) Repeat for $v + u = 1;52,30 = \frac{15}{8}$.

   (d) Repeat for $v + u = 2;05 = \frac{25}{12}$. This is line 9 of the tablet.

   (e) Repeat for $v + u = \frac{20}{9} = 2;13,20$.

9. Solve the following problem from tablet YBC 4652. The meaning of the problem isn't completely clear, so make your best guess.

   > I found a stone, but did not weigh it; after I subtracted one-seventh, added one-eleventh (of the difference), and then subtracted one-thirteenth (of the previous total), it weighed 1 *mina* (= 60 *gin*). What was the stone's weight?

10. Solve the following problem from tablet YBC 6967.

    A number exceeds its reciprocal by 7. Find the number and the reciprocal.

    In this case, two numbers are reciprocals if their product is 60.

11. For this question it is helpful to think about the corresponding facts for decimals.

    (a) Explain the observation on page 11 regarding which reciprocals n have a terminating sexagesimal. Can you prove this?

    (b) Find the periodic sexagesimal representation of $\frac{1}{7}$. Use geometric series to *prove* that you are correct.

12. The AM–GM inequality and some basic analysis (convergence of bounded monotone sequences) will be helpful for this question.

    (a) Suppose (a_n) is a positive sequence satisfying the recurrence $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$. Prove that $a_n \geq \sqrt{2}$ whenever $n \geq 2$.

    (b) Prove that $\lim a_n = \sqrt{2}$.

13. (a) The Babylonians used an approximation of the form $A \approx \frac{1}{12}c^2$ for the area of a circle in terms of its circumference. To what approximation of π does this correspond?

    (b) Use a Babylonian method to show that $\sqrt{3} \approx \frac{7}{4}$.

    (c) A 'bulls-eye' (pictured) is constructed using congruent circular arcs built from a circumscribed equilateral triangle.

    If the arc-length is a, use Babylonian approximations to prove that the area of the bulls-eye is $\approx \frac{9}{32}a^2$. What, approximately, are its dimensions (width/height)?
    (*Use as much modern trigonometry as you like!*)