3 Ancient Greek Mathematics

3.1 Overview of Ancient Greek Civilization & Early Philosophy

Ancient Greek civilization dates from around 800 BC, centered on the peninsula between the Adriatic Sea (to the west) and the Aegean (to the east) that forms the mainland of the modern Greek state. There was no central controlling empire. Semi-independent city-states were connected by trade and culture, but largely governed themselves. Accomplished seafarers, the Greeks extended their reach round the northern and eastern coasts of the Mediterranean, from Iberia (Spain) to the Black Sea, Anatolia (western Turkey), and Egypt, building and capturing a network of city-states and trading outposts.

Greek Territory around 500 BC

Philip of Macedon 'unified' the Greek peninsula just before his death in 336 BC. His son, Alexander the Great, subsequently launched a massive military campaign conquering Persia, Egypt, Babylon, and western India; he died in Babylon in 323 BC. Alexander left various governors to manage his conquered territories: some of these political structures endured for centuries (Egypt's Ptolemaic dynasty), whereas others were overthrown after only a few years (India/Pakistan). While Alexander's conquests did not produce a long-lasting centralized Greek empire, they were effective at expanding the reach of Greek culture and philosophy, and brought external ideas into the Greek tradition.

The core part of Greek territory was absorbed by the Roman empire around 146 BC. In line with typical Roman practice, those who agreed to accept Roman governors and taxation could eventually become Roman citizens.[7] As such, the Greek culture of inquiry and scholarship staid largely intact under Roman rule. Greek learning was central to the later Byzantine (eastern Roman) empire centered on Byzantium/Constantinople/Istanbul, which lasted until the capture

[7] While this might sound reasonable, resistance wasn't realistic, and a significant period of enslavement might first have to be endured...

of Constantinople by the Islamic Ottomans in 1453. For centuries prior, Islamic scholarship had itself been significantly influenced by the ancient Greeks. On consequence of the fall of Constantinople was the exodus of scholars to Rome, helping to fuel the nascent European Renaissance.

Early Greek Thought

Greek mathematics is part of a much wider philosophy encompassing a change of emphasis from practicality to abstraction. One reason for this was the Greek blending of religion/mysticism with natural philosophy: a desire to describe the natural world while preserving the (seeming) perfection and logic in the gods' design.

Early Greek inquiry into natural phenomena involved the personification of nature (e.g., sky = man, earth = woman). By 600 BC, philosophers were attempting to describe natural phenomena in terms of logical structures. For example, some viewed matter as being comprised of the 'four elements' (fire, earth, water, air) combined in the correct proportions. While the system of the world was considered divinely-designed, explanations relying on the whims of the gods came to be discouraged.

While the Greeks certainly used mathematics for practical purposes, philosophers idealized logic and were unhappy with approximations. This led to the development of *axiomatics, theorems* and *proof*, concepts for which there is scant pre-Greek evidence. The ancient Greek language is indeed the source of three words of critical importance:

Mathematics *Mathematos* (μαθήματος) meant knowledge or learning. The term covered essentially anything that might be taught in Greek schools. The stem *math* (knowledge) is preserved in other modern usage: e.g. *polymath.*

Geometry Literally *earth-measure*, this is a combination of two terms:

- *Gi* (γη) Dates from pre-5th century BC, meaning *land, earth* or *soil.* Capitalized (Γη) it could refer to the *Earth* (as a goddess).
- *Metron* (μέτρον) A *weight* or *measure*, a *dimension* (length, width, etc.), or the *metre* (rhythm) in music.

Theorem From *theoreo* (θεώρέω), meaning 'I contemplate/consider.' In a mathematical context this become *theorema* (θεώρημα): a proposition to be proved.

Greek Education and Scholarship

Ancient Greece had several schools, mostly private and open only to men. Typically arithmetic was taught until age 14, followed by geometry and astronomy until age 18.

The most famous scholars of ancient Greece were the Athenian trio of Socrates, Plato and Aristotle, each of whom taught his successor,[8] and whose writings became central to both Western and Islamic philosophical tradition. Plato's *Academy* in Athens provided the model for centuries of schooling; the centrality of geometry to Plato's curriculum was evidenced by the famous inscription above its entryway: "Let none ignorant of geometry enter here."

[8] The birth of Socrates to the death of Aristotle covered 470–322 BC.

Ancient Greek Enumeration

The Greeks had two primary forms of enumeration.

Attic Greek The Attic system (Attica refers to the area around Athens) dates from 7-800 BC. Strokes were used for 1–4, with larger numerals using the first letter of the words for 5, 10, 100, 1000 and 10000. For example,

- Πεντε (*pente*) means five, so Π denoted the number 5.
- Δεκα (*deca*) means ten, so Δ represented 10.
- Η (*hekaton*), Χ (*khilias*) and Μ (*murias*) denoted 100, 1000 and 10000 respectively.
- 𐅄, 𐅅 and 𐅆 represented 50, 500, and 5000 respectively.
- Combinations represented larger numbers: e.g., Χ𐅅ΗΗΠΙ= 1706. This is similar to both Egyptian hieroglyphs and Roman numerals. Unlike the latter, Attic Greek did not use pre-subtraction: thus 9 was written ΠΙΙΙΙ instead of ΙΔ.

In modern Greece, Attic numerals are still used occasionally for formal purposes, similarly to how other western countries might use Roman numerals (e.g. construction date of a building).

Ionic Greek Ionia refers to the Anatolian coast. This system dates from around 400 BC and uses the Greek alphabet, an approach possibly copied from Egyptian hieratic enumeration. Ionic numerals slowly displaced the older Attic system.

The Ionic alphabet included several unfamiliar symbols whose shapes and usage changed over time: *digamma* (Ϝ), *stigma* (ϛ), *qoppa* (ϙ, ϟ), and *sampi* (ͳ, ϡ).

Larger numbers were depicted in various ways. A left subscript mark like a comma denoted thousands and/or M (with superscripts) for multiples of 10000. For example,

$$35268 = {}_{,}\lambda{}_{,}\varepsilon\sigma\xi\eta = \overset{\gamma}{\mathrm{M}}{}_{,}\varepsilon\sigma\xi\eta$$

1	α	10	ι	100	ρ
2	β	20	κ	200	σ
3	γ	30	λ	300	τ
4	δ	40	μ	400	υ
5	ε	50	ν	500	φ
6	Ϝ, ϛ	60	ξ	600	χ
7	ζ	70	ο	700	ψ
8	η	80	π	800	ω
9	θ	90	ϙ, ϟ	900	ͳ, ϡ

The Greeks also employed Egyptian fractions, denoting reciprocals with a *keraia*: e.g., ¼ = δ′ along with a few special symbols like ∠′ for ½ (perhaps related to the special Egyptian hieroglyph for the same). The use of Egyptian fractions dominated in Europe well into the middle ages.

Both the Attic and Ionic systems were suitable for record-keeping but terrible for calculations! Later Greek mathematicians adapted the Babylonian sexagesimal system with Ionic notation, helping cement its modern use of in astronomy, navigation and time-keeping.
Ionic enumeration is still used in modern Greece, though Hindu–Arabic numerals are now more typical. By the middle-ages, a bar was placed over numerals to distinguish them from words, while modern practice is to insert a *keraia* similarly to how the ancients denoted fractions: e.g. $89 = \overline{\pi\theta} = \pi\theta'$.

Exercises 3.1. As with other ancient numerical systems, what matters is their structure (positional, decimal, etc.), not the details. Don't memorize the ancient Greek alphabet!

1. State the number 1896 in both Attic and Ionic notation.

2. Represent $\frac{8}{9}$ as a sum of distinct unit fractions (Egyptian style). Express the result in (Ionic) Greek notation.

   (*The answer to this problem is non-unique*)

3. For tax purposes, the ancient Greeks would approximate the area of a quadrilateral field by multiplying the averages of the two pairs of opposite sides. In one example, the two pairs of opposite sides were given as

$$a = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \quad \text{opposite} \quad c = \frac{1}{8} + \frac{1}{16}, \quad \text{and,}$$
$$b = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \quad \text{opposite} \quad d = 1$$

   where the lengths are in fractions of of a *schonion*, a measure of approximately 150 feet. Find the average of a and c, the average of b and d, and thus the approximate area of the field in square *schonion.* The taxman then rounds up the answer to collect a little more tax!

3.2 Pre-Euclidean Greek Mathematics

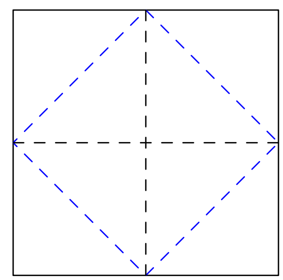
By subsuming much that came before it, the publication of Euclid's *Elements* (c. 300 BC) forms a natural breakpoint in Greek mathematics. In this section, we consider the contributions of several pre-Euclidean mathematicians. Very few sources exist from before 400 BC, so almost everything is inferred from the writings and commentaries of others.[9]

Thales of Miletus (c. 624–546 BC)

Thales is credited as one of the first people to state general abstract principles. A trader based in Miletus, a city-state in Anatolia, Thales travelled widely and was likely exposed to mathematical and philosophical ideas from all round the Mediterranean. Here are several statements at least partly attributed to him:

- The angles at the base of an isosceles triangle are equal.
- Any circle is bisected by its diameter.
- A triangle inscribed in a semi-circle is right-angled (still known as Thales' Theorem).

Thales' major development is *generality*: his propositions concern *all* triangles, *all* circles, etc. Babylonian & Egyptian mathematics had *examples*, but there is scant indication of abstract generality. In line with the Greek idea of a theorem ('to look at'), Thales' reasoning was likely pictorial.



As an example of contemporary geometric reasoning, by 425 BC Socrates could describe how to halve (or double) the area of a square by joining the midpoints of edges and folding.

Pythagoras of Samos (c. 572–497 BC)

Samos is an island in the Aegean, not far from Miletus. Like Thales, Pythagoras also travelled widely, eventually settling in Croton, southern Italy, where he founded a school—this endured a century after his death. Plato is believed to have learned much of his mathematics from a Pythagorean named Archytas.

The Pythagoreans practiced a mini-religion whose ideas lay outside the mainstream of Greek society.[10] One of their mottos, "All is number," emphasised their belief in the centrality of pattern and proportion. The following quote[11] gives some flavor of the Pythagorean way of life.

> After a testing period and after rigorous selection, the initiates of this order were allowed to hear the voice of the Master [Pythagoras] behind a curtain; but only after some years, when their souls had been further purified by music and by living

[9]For instance, most of our knowledge of Socrates comes from the voluminous writings of Plato and Aristotle. The earliest known Greek textbook/compilation was written around 430 BC (*Elements of Geometry* by Hippocrates of Chios); no copy survives, though most of its material probably made it into Book I of Euclid.

[10]They believed in the transmigration of souls, were vegetarians, and accepted women as students; controversial ideas indeed!

[11]Van der Waerden, *Science Awakening* pp 92–93.

in purity in accordance with the regulations, were they allowed to see him. This purification and the initiation into the mysteries of harmony and of numbers would enable the soul to approach [become] the Divine and thus escape the circular chain of re-births.

Musical harmony The relationship between music and number was of great interest to the Pythagoreans. They are credited with relating musical intervals to ratios of lengths of vibrating strings:

- Identical strings whose lengths are in the ratio 2:1 vibrate an *octave* apart.
- A *perfect fifth* corresponds to the ratio 3:2.
- A *perfect fourth* corresponds to the ratio 4:3.

The use of these ratios to tune musical instruments is still known as *Pythagorean tuning.*[12]

Mathematical results attributable to the Pythagoreans Theorems 21–34 in Book IX of Euclid's *Elements* are Pythagorean in origin. For instance:

Theorem. **(IX.21)** *A sum of even numbers is even.*

(IX.27) *Odd less odd is even.*

The Pythagoreans also studied perfect numbers: those which equal the sum of their proper divisors (e.g., $6 = 1 + 2 + 3$). They seem to have observed the following famous result.

Theorem (IX. 36). *If* $2^n - 1$ *is prime then* $2^{n-1}(2^n - 1)$ *is perfect.*

They also considered square and triangular numbers ($\frac{1}{2}m(m+1)$) and tried to express geometric shapes as numbers; all of this was in line with their belief that all matter could be formed from basic shapes.

Of course, the most famous result associated with Pythagoras is the theorem that bears his name.

Theorem (I. 47). *The square on the hypotenuse of a right-triangle equals (has the same area as) the sum of the squares on the remaining sides.*

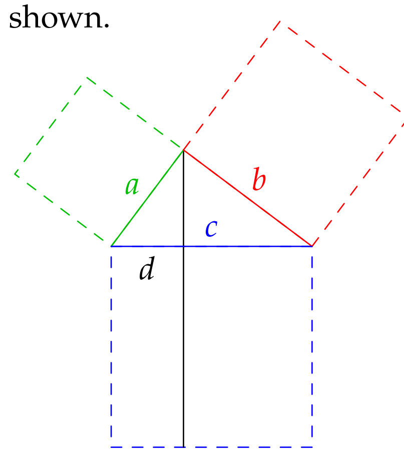
This result is the capstone of Book I or Euclid's *Elements*; indeed the book seems to have been structured precisely to provide a rigorous justification. It is generally believed that the Pythagoreans did not possess a rigorous proof, though here is an inferred argument that would have been in line with their approach.

[12] This contrasts with the more modern *equal temperamnent* tuning, where the frequency ratios of all semitone intervals are identical (twelfth-root of 2). Equal temperament is mathematically challenging because it requires the accurate computation of this ratio.

Inferred proof. Label the side-lengths of the right-triangle a, b, c, as shown.
Drop the altitude to the hypotenuse c.
Let d be the length of the part of the hypotenuse underneath a.
The large triangle comprises two smaller similar triangles, whecne

$$a : d = c : a \implies a^2 : ad = cd : ad$$
$$\implies a^2 = cd$$

The square on a therefore has the same area as the rectangle below the left side of the hypotenuse.
Repeat the calculation on the other side to obtain $b^2 = c(c-d)$, and sum to complete the proof.



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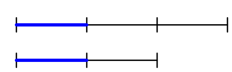
From a modern viewpoint, there is nothing wrong with this argument. So why did Euclid feel the need to devote an entire book to re-proving it?! The problem lies with the Pythagoreans' contradictory use of the ancient Greek concepts of number and length...

Ratios and Incommensurability: length versus number

As with other ancient cultures, *number* to the Greeks meant *positive integer.* While the Greeks certainly used fractions, these were properly considered ratios of integers. Indeed ratios were used to *compare* physical quantities such as length and volume, as well as non-physical quantities like number. For instance, we'd happily say that the integers 8 and 4 are in the ratio 2 : 1.

Definition. Let m, n be positive integers. Lengths (or other physical quantities) are said to be in the ratio $m : n$ if some sub-length divides exactly m times into the first and n times into the second.
Lengths are *commensurable* if some sub-length divides exactly into both, and *incommensurable* otherwise.

For instance, the pictured rods have commensurable lengths in the ratio $\ell_1 : \ell_2 = 3 : 2$, since the largest possible common sub-length goes three times into the first rod and twice into the second.



While modern mathematics has no problem with *irrational ratios* (e.g., the diagonal of a square to its side is $\sqrt{2} : 1$), this conflicts with the Pythagorean belief that *any two lengths were commensurable.* Identifying lengths with real numbers (underlined), we restate this assertion in modern language:

$$\forall \underline{m}, \underline{n} \in \mathbb{R}^+,\ \exists \underline{\ell} \in \mathbb{R}^+,\ a, b \in \mathbb{N},\ \text{such that } \underline{m} = a\underline{\ell} \text{ and } \underline{n} = b\underline{\ell}$$

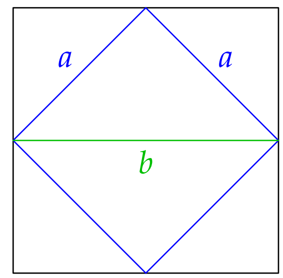
But this is utter nonsense, for it claims that every ratio of real numbers $\underline{m} : \underline{n} = a : b$ is *rational*!
The Pythagorean commensurability supposition stems from their basic tenets: all is number (including length ratios), and the design of the gods is perfect (number means positive integer).

The discovery of incommensurable ratios produced something of a crisis for the Pythagoreans; a possibly apocryphal story states that a disciple named Hippasus (c. 500 BC) was set adrift at sea as punishment for its revelation.

By 340 BC, the Greeks were happy to state that incommensurable lengths exist.

Theorem (Aristotle). *If the side and diagonal of a square are commensurable, then odds equal evens.*

Inferred proof. In Socrates' doubled-square, suppose that side and diagonal of the smaller square are in the ratio $a : b$; note that a and b are integers!



Assume (without loss of generality) at least one of the integers is odd, for otherwise the common sub-length may be doubled. The larger square is twice the area of the smaller, whence the square numbers have ratio

$$b^2 : a^2 = 2 : 1$$

Plainly b^2 is even and so also is b. Writing $b = 2k$, we have

$$4k^2 : a^2 = 2 : 1 \implies a^2 : k^2 = 2 : 1$$

whence a^2, and thus a, is also even. Whichever of a, b was odd is also even: contradiction! ∎

Note the similarity of this argument to the modern proof of the irrationality of $\sqrt{2}$.

This discussion helps to illuminate the importance of geometry to the Greeks. Lengths could be anything (in essence they are continuous), whereas numbers are more limited: numbers cannot describe the relationship between all lengths.

Returning to the Pythagoreans' inferred proof of their own theorem, in the ancient Greek context, the argument only makes sense if the numbers a, b, c, d are *integer multiples of a common sub-length*. Aristotle's result shows that such a sub-length need not exist.

Theaetetus of Athens (417–369 BC) Likely the source of much of the most difficult parts (Books VII & X) of the *Elements*, Theaetetus' approach to (in)commensurability comes from applying what is now known as the Euclidean algorithm to lengths (line segments).

Definition. Repeatedly apply the division algorithm to lengths $\underline{a} > \underline{b}$ ($\underline{a}$ is longer than $\underline{b}$):

$$\begin{array}{lll} \underline{a} = q_1\underline{b} + \underline{r}_1 & \underline{r}_1 < \underline{b} & (\exists q_1 \in \mathbb{N}_0 \text{ and a length } \underline{r}_1 < \underline{b}) \\ \underline{b} = q_2\underline{r}_1 + \underline{r}_2 & \underline{r}_2 < \underline{r}_1 & \\ \underline{r}_1 = q_3\underline{r}_2 + \underline{r}_3 & \underline{r}_3 < \underline{r}_2, \quad \text{etc.} & \end{array}$$

We say that $\underline{a}$ and $\underline{b}$ are *commensurable* if the algorithm terminates: some remainder $\underline{r}_n$ divides exactly into $\underline{r}_{n-1}$. Otherwise $\underline{a}$ and $\underline{b}$ are *incommensurable.*

Ratios are *equal* $\underline{a} : \underline{b} = \underline{c} : \underline{d}$ precisely when the sequences of quotients in the algorithm are equal.

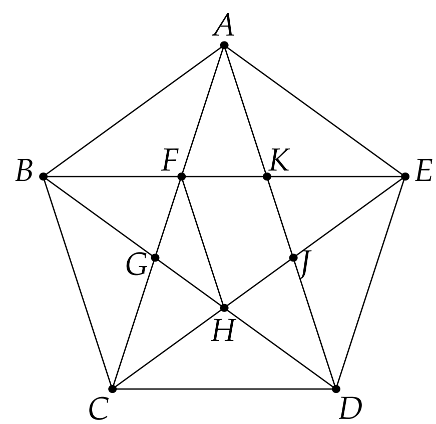
If $\underline{a}$ and $\underline{b}$ are commensurable, then $\underline{r}_n$ is their *greatest common sub-length*. If we write $\underline{a} = a\underline{r}_n$ and $\underline{b} = b\underline{r}_n$ for some integers a, b and rewrite the algorithm in modern fashion, we obtain the standard Euclidean algorithm demonstration of $\gcd(a, b) = 1$.

Example 1 The algorithm applies to pairs of integers (start with commensurable rods!).

$$\begin{aligned} \underline{37} &= 2 \cdot \underline{13} + \underline{11} \\ \underline{13} &= 1 \cdot \underline{11} + \underline{2} \\ \underline{11} &= 5 \cdot \underline{2} + \underline{1} \\ \underline{2} &= 2 \cdot \underline{1} \end{aligned} \qquad \begin{aligned} \underline{148} &= 2 \cdot \underline{52} + \underline{44} \\ \underline{52} &= 1 \cdot \underline{44} + \underline{8} \\ \underline{44} &= 5 \cdot \underline{8} + \underline{4} \\ \underline{8} &= 2 \cdot \underline{4} \end{aligned}$$

Since we obtain the same sequence of quotients $(2, 1, 5, 2)$, we conclude that $37 : 13 = 148 : 52$.

Example 2 We outline an argument showing that the diagonal $\underline{a} = \overline{AC}$ and side $\underline{b} = \overline{AB}$ of a regular pentagon $ABCDE$ are incommensurable.



1. $\triangle BAG$ is isosceles (count angles!), whence $|AB| = |AG|$.
2. The first line of the algorithm reads
$$|AC| = |AG| + |GC| = |AB| + |GC|$$
so we write $\underline{a} = q_1\underline{b} + \underline{r}_1$ where $r_1 = |GC|$ and $q_1 = 1$.
3. Since $|GC| = |AF|$, the second line of the algorithm reads
$$|AB| \big(= |AG| \big) = |AF| + |FG| = |GC| + |FG|$$
so that we again have quotient $q_2 = 1$.
4. Prove that $\triangle DCG \cong \triangle EHF$. But then $|GC| = |FH|$ is the diagonal of the interior regular pentagon. The third line of the algorithm ($|GC|$ divided by $|FG|$) is therefore the same as the first: dividing the diagonal to the side of a regular pentagon. The algorithm therefore continues forever with all quotients being 1.

Example 3 In modern language, the diagonal to the side of a square is the incommensurable ratio $\sqrt{2} : 1$. We apply Theaetetus' algorithm:

$$\begin{aligned} \underline{\sqrt{2}} &= 1 \cdot \underline{1} + (\underline{\sqrt{2} - 1}) \\ \underline{1} &= 2 \cdot (\underline{\sqrt{2} - 1}) + (\underline{3 - 2\sqrt{2}}) \end{aligned}$$

Observe that $3 - 2\sqrt{2} = (\sqrt{2} - 1)^2$, whence the second line reads $1 = 2x + x^2$. The following lines in the algorithm may therefore be obtained by repeatedly multiplying by x:

$$x = 2x^2 + x^3, \quad x^2 = 2x^3 + x^4, \quad \text{etc.,}$$

which produces a never-ending sequence[13] of quotients: $1, 2, 2, 2, 2, 2, \ldots$

[13] If you like number theory, investigate the relationship of Theaetetus' algorithm to continued fractions...

Eudoxus of Knidos (c. 390–337 BC) One of the most prolific pre-Euclidean mathematicians, Eudoxus attended and perhaps taught at Plato's academy. He is famous for explaining how to calculate with ratios of lengths (segments).

Definition. $A : B > C : D$ if there exist positive integers m, n such that $mA > nB$ and $mC \leq nD$.

At first glance it appears as if Eudoxus is telling us how to compare *rational* numbers. Indeed, if A, B, C, D are integers, then the definition may be rewritten:

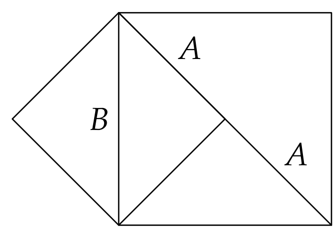
$$\frac{A}{B} > \frac{n}{m} \geq \frac{C}{D}$$

is trivially satisfied by taking $m = D$ and $n = C$. To Eudoxus, however, A, B, C, D could be interpreted as *segments*. Building on the work of Theaetetus, his mathematics told the Greeks how to approximate incommensurable ratios with rational ones.

Example 1 To see that $13 : 3 > 17 : 4$, simply choose $m = 4$ and $n = 17$ to obtain

$$4 \cdot 13 = 52 > 51 = 3 \cdot 17$$

Example 2 We show that the ratio of the side to the diagonal of a square is greater than $1 : 2$; equivalently, the diagonal is less than twice the side.[14]



As pictured, side : diagonal = $A : B$. Choosing $m = D = 2$ and $n = C = 1$, we see that

$$mA = 2A > B = nB \qquad \text{(diag > side of large square)}$$

In modern language, this is merely $\frac{1}{\sqrt{2}} > \frac{1}{2}$.

Arguments about Infinity: Zeno of Elea (c. 450 BC)

The Greeks provided some of the first discussions (and disagreements) about infinity and infinitesimals. Zeno's arguments, in particular, have provided philosophical fodder for millennia, illustrating some of the difficulties behind modern calculus. Here are two of the most famous:

Achilles and Tortoise Achilles chases a Tortoise. After time t_0, Achilles reaches the Tortoise's starting position but the Tortoise has moved on. After a second time interval t_1, Achilles reaches the Tortoise's second position; again the Tortoise has moved. In this manner Achilles spends a total time $t_0 + t_1 + t_2 + \cdots$ chasing the Tortoise. Zeno's paradoxical conclusion is that Achilles never catches the Tortoise.

This paradox may be resolved (see Exercise 8), at least if we assume both runners travel at constant speeds. Though it be split into infinitely many subintervals of time, the total duration of the chase can expressed as the (finite) value of a convergent *infinite series.*

[14]This is trivial by the triangle inequality, though it is interesting to see it in Eudoxus' context.

Arrow paradox An arrow is shot from a bow. At any instant the arrow doesn't move. If time is made up of instants, then the arrow never moves.

This time Zeno debates the the idea that a positive time interval can be considered as a sum of infinitesimal instants. This is the essential philosophical difficulty of *integration.*

Zeno's arguments were marshaled against Galileo (early 1600s) and both Newton and Leibniz (late 1600s/early 1700s) as their methods of infinitesimal calculus became widely used.

Constructions and Geometry

By the mid 5th century BC, Greek mathematicians were solving geometric problems using *ruler-and-compass* (peg-and-cord) constructions. This approach might have come to Greece from India, or possibly arose organically. Constructions were based on three rules, which became the first three postulates (axioms) of Book I of Euclid's *Elements.*

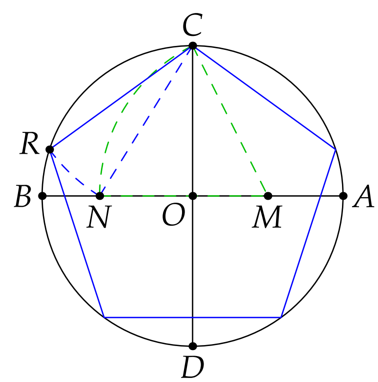
1. Two points may be joined by a straight line (segment).
2. A segment may be extended indefinitely.
3. Given a center and radius, one may draw a circle.

Theorems were often stated as *problems*: e.g., *to bisect a given angle.* A proof provided first a *construction,* then an argument justifying that the construction really had solved the problem.

By the time of Euclid, the Greeks knew many constructions, including those of an equilateral triangle, a square and a regular pentagon in a given circle.

Example We construct a regular pentagon in a given circle.[15]

1. Draw perpendicular diameters $\overline{AB}$, $\overline{CD}$ and bisect $\overline{OA}$ at M.
2. Draw an arc centered at M with radius $|CM|$. Let N be the intersection of this arc with $\overline{OB}$.
3. Draw an arc centered at C with radius $|CN|$. Let R be the intersection of this arc with the original circle.
4. Move $|CR|$ around the circle to create a regular pentagon.



A purely geometric proof of the validity of this construction is too difficult for us, but you can easily check it using a calculator, Pythagoras' and the cosine rule: if the circle has radius 2, then

$$\begin{aligned}|CR|^2 = |CN|^2 &= |ON|^2 + |OC|^2 \\ &= (\sqrt{5}-1)^2 + 2^2 = 10 - 2\sqrt{5} \\ &= 2^2 + 2^2 - 2\cdot 2\cdot 2\cos 72^\circ\end{aligned}$$

[15] Theorem IV. 11 of the *Elements* presents a less practical construction. Ours follows from Theorem XIII. 10: if a regular pentagon, hexagon and decagon are inscribed in a circle, then their sides form a right-triangle.

The appearance of $\sqrt{5}$ in this discussion relates to the golden ratio $\frac{1+\sqrt{5}}{2}$ which is in fact the ratio of the diagonal to the side of the pentagon: exactly what Theaetetus was calculating on page 27. Pentagons and pentagrams were viewed as mystical by many ancient Greeks.

Construction problems have motivated mathematicians ever since. In 1796, Carl Friedrich Gauss (then only nineteen) constructed a regular 17-gon. A full classification of the constructable regular polygons came in 1837.

Theorem. *A regular n-gon is constructable (using only ruler-and-compass constructions) if and only if* $n = 2^k F_1 \cdots F_r$ *where* $F_1, \ldots, F_r$ *are distinct primes of the form* $2^{(2^n)} + 1$.

After the 17-gon, the next prime-sided constructable n-gon has $257 = 2^{2^3} + 1$ sides!

By 400 BC, the Greeks were referencing the second and third *impossible constructions of antiquity*:

1. Trisecting a general angle.
2. Doubling (the volume of) a given cube.
3. Squaring a circle (construct a square with the same area as a given circle).[16]

It wasn't until the advent of field theory in the 1800s that such were proved to be impossible using ruler-and-compass constructions.

Summary

Several of the mathematical techniques in this section are difficult and the results technical. It isn't important to become proficient with all of these ideas. Play with them to help you develop an appreciation of two overarching points:

1. Even before Euclid, the focus of Greek mathematics was more abstract and less practical than that of other ancient cultures (Egypt, Babylon, China, etc.), largely due to the influence of wider Greek philosophy and religion. The modern liberal arts ideal of learning for its own sake—to celebrate the beauty of knowledge and expand the mind—is, to a large extent, a Greek inheritance.
2. The ancient Greeks pondered fundamental mathematical questions and concepts—*number* versus *length*, continuity, irrationality, infinitesimals, constructions—ideas that have stimulated mathematical research ever since. These particular issues would not be resolved rigorously until the 1800s, when luminaries such as Gauss, Cauchy and Riemann developed modern analysis and algebra.

[16]"You can't square that circle," is a metaphor for something impossible, or beyond human comprehension.

Exercises 3.2. *Key concepts: Abstraction, Ratios of magnitudes (number ≠ length), Incommensurability (irrationality), Difficulties reasoning with infinity, Ruler-and-compass constructions*

1. Construct five Pythagorean triples using the formula $\left(n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right)$ where n is odd. Construct five more using the formula $\left(m, \left(\frac{m}{2}\right)^2 - 1, \left(\frac{m}{2}\right)^2 + 1\right)$ where m is even.
2. Suppose $2^n - 1 = p$ is prime (its only positive divisors are itself and 1). List the positive divisors of $2^{n-1}(2^n - 1)$ and hence prove Theorem IX. 36.
3. Draw a *picture* with dots to show that eight times a triangular number plus 1 makes a square, and that any odd square minus 1 is eight times a triangular number. That is:
   (a) $8 \cdot \frac{1}{2}m(m+1) + 1$ is a perfect square.
   (b) If n is odd, then $n^2 - 1 = 8 \cdot \frac{1}{2}m(m+1)$ for some m.
4. Find a ruler-and-compass construction to bisect a given angle and show that it is correct.
5. Sketch a construction inscribing a regular hexagon in a circle.
   (*Assume you can construct an equilateral triangle on a given segment—Thm I. 1 of Euclid*)
6. (A line-doubling paradox) One line has twice the length of another and so has *more* points. However, there is a bijective correspondence between the points on these lines; the two lines therefore have the *same number* of points.
   Explain the second observation. How can you resolve the paradox?
7. The *cycle of fifths* is a musical concept stating that twelve perfect fifths equals seven octaves (pg. 24). State this claim *numerically,* and show that it is a contradiction.
   (*Hint: two strings are seven octaves apart if their lengths are in the ratio* $2^7 : 1$)
8. Recall Zeno's paradox of Achilles and the Tortoise. Suppose Achilles travels at speed v_A, the Tortoise at speed $v_B < v_A$, and that the tortoise starts a distance d ahead of Achilles.
   (a) Prove that $t_n = \frac{d}{v_A}\left(\frac{v_T}{v_A}\right)^n$ for each positive integer n.
   (b) Compute $\sum_{n=0}^{\infty} t_n$ using the geometric series formula from calculus.
   (c) Verify the time-value computed in (b) by considering the motion of Achilles relative to the Tortoise.
9. Use Theaetetus' definition of equal ratios to prove that $46 : 6 = 23 : 3$.
10. (Hard) A line of length 1 is divided at x so that $\frac{1}{x} = \frac{x}{1-x}$. Prove that 1 and x are incommensurable. Moreover, show that $1 : x$ is *the same* as diagonal : side of a regular pentagon.
    (*Hint: the first line of the algorithm is* $1 = x + x^2 \ldots$)
11. (Hard) Let $a > b$ and c be *positive lengths.* Use Eudoxus' definition to *prove* that $c : b > c : a$.
    (*Hint: let* n *be the smallest integer such that* $n(a-b) \geq c$*; its existence is the "archimedean property"*)

3.3 Euclid and the *Elements*

The city of Alexandria was founded after the Greek general Alexander the Great conquered Egypt in 323 BC.[17] Alexandria's Library was constructed around 320 BC as a means of organizing the knowledge of the world and as a demonstration of Greek power. Although it was seriously damaged on several occasions, the Library remained a center of scholarship until around AD 500. To the right is a map showing Alexandria's location in modern Egypt. Below is the city around AD 400: note the size and centrality of the Library.

It is generally believed that Euclid lived in Alexandria and worked its Library in the years after its founding. As with any ancient figure, it is hard to be precise about Euclid the person: his age, social status, or exact position at the Library. He might personally have written, transcribed and understood all of the mathematics that bears his name, though it is perhaps more likely that he was the leader or manager of a group of scholars.[18]

It is hard, however, to argue against his *Elements* (c. 300 BC) as the most influential mathematical text ever produced. A compilation and organization of earlier mathematical work rather than a pure original, it was edited and added to over the centuries, eclipsing and subsuming other works. Particularly important were the edits of Theon of Alexandria and his daughter Hypatia (c. AD 400), both prolific scholars in their own right. Due to edits such as these, the precise contents of the original are unknown.

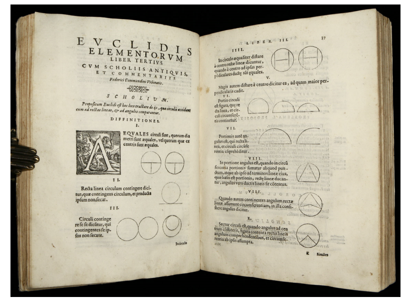
[17] Alexander is estimated to have founded approximately ten cities, naming most of them for himself!

[18] It is hardly surprising that we know so little about someone who lived 2300 years ago. Similar questions are sometimes raised about William Shakespeare, who lived only 400 years ago! Euclid is also credited with writing other mathematical works beyond the *Elements*, including the *Data*. Such prolific output lends credence to the idea that "Euclid the Mathematician" was at least partly a collective endeavor.

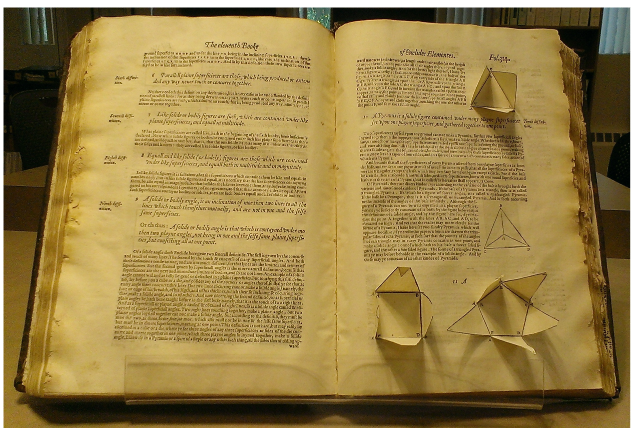
Extant fragments of the *Elements* date to around AD 100—*four hundred years* after Euclid. The earliest (almost) complete copy is from the the 9th century. This copy is written in Greek and resides at the Vatican. It is also missing some of the edits of Theon & Hypatia, thus demonstrating that multiple versions were in circulation. Over the centuries, many editions and variations have been produced, some of which are pictured here.

Earliest Fragment c. AD 100

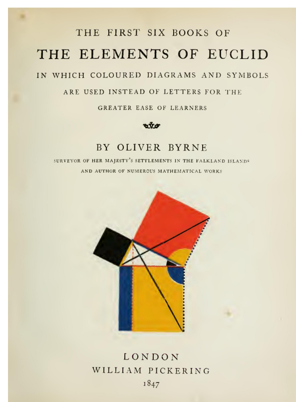
Full copy, Vatican 9th C



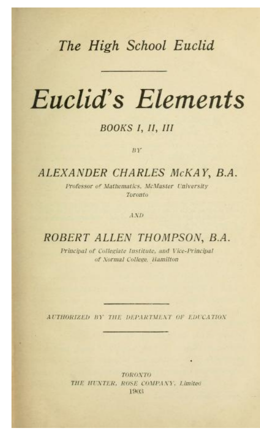
Latin translation, 1572



Pop-up edition, 1500s



Color edition, 1847



School textbook, 1903

You can download Byrne's color edition here (very large file!). It is notably different from many earlier editions in that it contains a much longer list of definitions, inserts many more axioms, and relabels propositions 4 and 5 as axioms (pages xviii–xxiii). The picture on the cover page is Euclid's proof of Pythagoras' Theorem (Book I, Thm. 47).
The *Elements* has been central to western and middle-eastern education for centuries. Until the mid 20th century, high-school students would have learned directly from a "Euclid." Much of the reason for this is the logical presentation: people studied Euclid to train themselves to become more logical.

A Brief Overview of the *Elements*

The *Elements* consists of thirteen books covering two- and three-dimensional geometry, computations and number theory. One of its key features is its *axiomatic* presentation. Each book begins with a list of axioms, postulates and definitions, before proceeding to prove theorems deduced from these. This *axiomatic method* is essentially universal in modern mathematics and its advent is fundamentally what sets Greek mathematics apart from everything that came before.

We briefly discuss Book I, then give some flavor of the remainder of the text with a few example results. Several examples of material from later books were mentioned in the previous section.

Book I After the axioms/postulates come 48 theorems, culminating in a proof of Pythagoras' Theorem and its converse. It seems likely that Euclid organized Book I with the goal of rigorously proving this important result: recall (pg. 25) how the Pythagorean 'proof' relied on the erroneous notion of commensurability.

Here are the postulates from Book I: the first three are what define ruler-and-compass constructions (pg. 29).

- P1 Given any two points, a straight line can be drawn between them
- P2 Any line may be indefinitely extended
- P3 Given a center and a radius, a circle may be drawn
- P4 All right angles are equal to each other
- P5 If a straight line crosses two others so that the angles on the same side make less than two right angles, then the two lines meet on that side of the original.

The fifth postulate is awkwardly phrased. Here is an equivalent statement more accessible to modern readers:

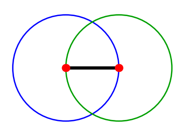
> *Playfair's axiom* : At most one parallel passes through a given point not on a line.

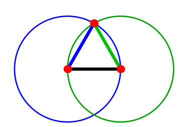
For centuries, mathematicians tried to prove that the parallel postulate was a theorem of the others. In the 1800s, with the advent of hyperbolic geometry, it was eventually shown that the parallel postulate is a necessary inclusion. Euclid's refusal to use the parallel postulate until Theorem 29 suggests he well understood this difficulty.

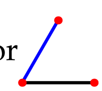
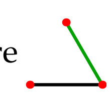
Theorems are often presented as *problems*. For instance, here is Euclid's first theorem.

Theorem (I.1). *Problem: To construct an equilateral triangle on a given segment.*

Euclid first provides a (ruler-and-compass) construction, before proving that his construction really solves the problem.

Proof. Given a segment  construct two circles (P3) 

Join one of the circle intersections to both endpoints of the segment (P1) 

We claim that the result is an equilateral triangle. Indeed the three sides are congruent, for  are radii of a common circle, as are  ∎

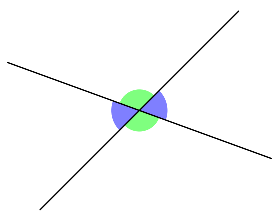
After this Euclid proceeds to establish several well-known results. Since this isn't a geometry class, we'll omit most of the details—what matters is the flavor. You can find much more detail here, in Byrne, or elsewhere.

Thm I. 4 Side angle side: if triangles have two pairs of congruent sides and the angles between them are also congruent, then the remaining sides and angles are congruent.

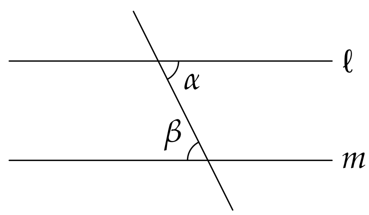
Thm I. 15 Vertical angles: if two lines meet, then the opposite angles made are congruent.

Thms I. 27 & 29 Suppose a line falls on two other lines ℓ, m making alternate angles α, β. Then:

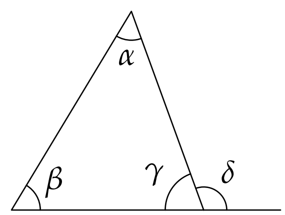
$$\alpha \cong \beta \iff \ell \parallel m \qquad \text{(I. 27 is } (\Rightarrow)\text{, I. 29 its converse)}$$



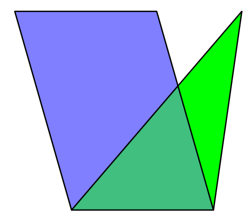
Thm I. 15



Thms I. 27/29



Thm I. 32



Thm I. 41

Thm I. 32 Angle sums in a triangle: if one side of a triangle is protruded, the exterior angle equals the sum of the opposite interior angles.

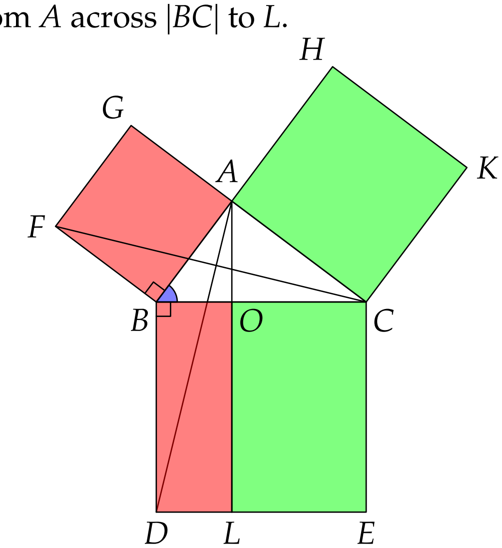
In the picture $\alpha + \beta = \delta$; in modern language $\alpha + \beta + \gamma = 180°$.

Thm I. 41 A parallelogram and a triangle on the same base and with the same height have area in the ratio 2:1.

The last two results of Book I are Pythagoras and its converse.

Theorem (I. 47). *The square on the hypotenuse of a right-triangle has area equal to the sum of the areas of the squares on the remaining sides.*

Proof. Given a right-angle at A, drop the perpendicular from A across $|BC|$ to L.



$\triangle FBC$ and $\square ABFG$ share the same base $\overline{BF}$ and height $\overline{AB}$. By Thm I. 41,

$$\text{area}(\square ABFG) = 2\,\text{area}(\triangle BCF)$$

Similarly (base $\overline{BD}$, height $\overline{BO}$)

$$\text{area}(\square BOLD) = 2\,\text{area}(\triangle ABD)$$

Side-Angle-Side (Thm I. 4) $\implies$ $\triangle ABD \cong \triangle FBC$; the triangles have the same area, and so

$$\text{area}(\square ABFG) = \text{area}(\square BOLD)$$

Similarly $\text{area}(\square ACKH) = \text{area}(\square OCEL)$. ∎

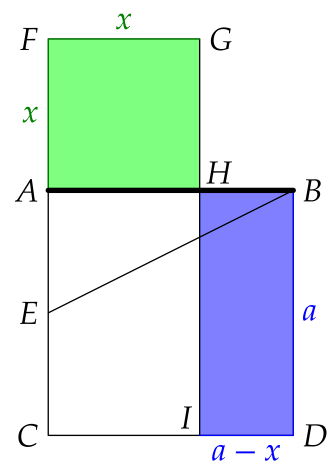
The converse (Thm I. 48) is Exercise 3.

Book II Geometric solutions to problems that would now be treated using algebra. Much of this is attributable to the Pythagoreans.

(Thm II. 11) A segment may be divided so that the rectangle contained by the whole and one of the sub-segments is equal to the square on the remaining sub-segment.

We rephrase this in modern language. Suppose the given segment $\overline{AB}$ has length a, our goal is to find H on $\overline{AB}$ such that $|AH| = x$ and $x^2 = a(a-x)$. Euclid is providing a *geometric solution* to a quadratic equation!

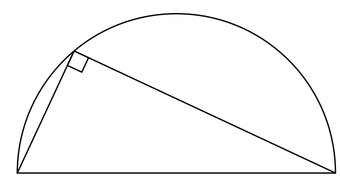
1. Construct $\square ABDC$ on $\overline{AB}$
2. Let E be the midpoint of $\overline{AC}$ and connect $\overline{EB}$
3. Extend $\overline{AC}$ and lay off $\overline{EF} \cong \overline{EB}$ on $\overline{AC}$ extended
4. Construct $\square AFGH$ and drop the perpendicular to I
5. To finish: prove that $\text{area}(\square AFGH) = \text{area}(\square BDIH)$



The solution $x = |AH| = \frac{\sqrt{5}-1}{2}a$ means that $\overline{AH} : \overline{HB}$ is the *golden ratio.*

Book III Theorems regarding circles and tangency. Most of this material likely came from Hippocrates.

(Thm III. 31) Thales' Thm: a triangle in a semi-circle is right-angled.



Book IV Construction of regular 3, 4, 5, 6 and 15-sided polygons inscribing and exscribing a circle. Also likely the work of Hippocrates.

Book V Ratios and magnitudes à la Eudoxus.

(Thm V. 11) If $a : b = c : d$ and $c : d = e : f$ then $a : b = e : f$

Book VI Ratios of magnitudes applied to geometry (similarity results).

(Thm VI. 4) Triangles with equal angles have corresponding sides proportional.

(Thm VI. 8) The altitude from the right angle of a right triangle divides it into two triangles similar to each other and the the original.

(Thm VI. 31) Corrected Pythagorean proof (pg. 25) of Pythagoras' using the proportions of Eudoxus and Thm VI. 8.

Book VII Divisibility and the Euclidean algorithm. Probably due to the Pythagoreans and Theaetetus.

Book VIII Number progressions, geometric sequences. Possibly due to studies in music by Archytas (a Pythagorean who taught Plato mathematics).

Book IX Number Theory: even/odd + perfect numbers.

(Thm IX. 20) There are infinitely many primes.

Book X Discussion of commensurable and incommensurable ratios. Long and difficult, possibly derived from Theaetetus.

Book XI Solid geometry (lines/planes in 3D).

(Thm XI. 28) A parallelepiped is bisected by its diagonal plane.

Book XII Ratios of areas and volumes (Eudoxus).

(Thm XII. 2) The areas of circles are in the same ratio as the squares on their diameters.

Book XIII Construction of regular polyhedra inside a sphere and their classification.

(Thm XIII. 10) If a regular pentagon, hexagon and decagon are inscribed in the same circle, then their sides form a right-triangle.

The details in each book are not what is important for us, though it is certainly interesting to see how many ideas familiar in modern mathematics are covered. Instead consider two final thoughts.

1. While Euclid covers a variety of topics, the Pythagorean difficulty of marrying length and number is central. The Greeks wanted to describe the natural (geometric) world using the perfection of numbers (positive integer). As such, their reasoning collided with the foundational mathematical problems of irrationality and continuity, issues that would not be satisfactorily resolved for thousands of years.

2. The *Elements*' central position in the (historical) education system is arguably more due to its axiomatic and logical presentation that its mathematical content. For over two millenia, people have studied Euclid to train themselves to make better, more rigorous arguments. Despite having little use for foundational geometry and number theory, many a trainee priest or lawyer[19] have honed their argumentative skills through the reading of Euclid's *Elements*.

One could study the *Elements* and its influence for a lifetime and not be done! Hopefully this very brief overview partly convinces you why the book had such a profound impact on mathematics.

[19]Famously including President Abraham Lincoln:

> "You can never make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight."

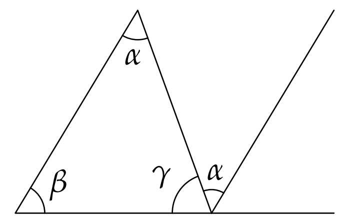
Exercises 3.3. *Key concepts: Axiomatic method, Logical structure, Influence on Education, Constructive (ruler-and-compass) proofs, Pythagoras without incommensurables, Focus on geometry and number theory*

1. Prove Thales' Theorem (III. 31) (pg. 36).

   (*Hint: start by joining the center of the circle to the apex of the triangle…*)

2. Use the picture to provide a proof of Thm I. 32: the sum of the three interior angles of a triangle is equal to two right-angles.

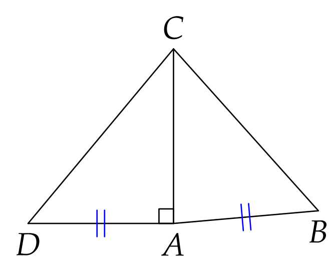
   Show that the proof depends on Thm I. 29, and therefore on the parallel postulate.
   (*This isn't quite the same as Euclid's argument*)

3. Suppose that the square on side $\overline{BC}$ of $\triangle ABC$ has the same area as the sum of the squares on the other sides $\overline{AB}$, $\overline{AC}$. As in the picture, draw a perpendicular $\overline{AD} \cong \overline{AB}$.

   (a) Explain why $\overline{DC} \cong \overline{BC}$.

   (b) Hence conclude that $\triangle ABC$ is right-angled at A.

4. Prove Thm III. 3: A diameter of a circle bisects a chord if and only if it is perpendicular to the chord.

5. Verify that Euclid's construction for Thm II. 11 really does solve the given problem.

   (*You can use modern algebra!*)

6. Draw a semi-circle with diameter $9 + 5 = 14$. Solve the equation $\frac{9}{x} = \frac{x}{5}$ *geometrically*, by constructing a vertical line whose length is x.

7. Show that areas of similar segments of circles are proportional to the squares of the length of their chords.

   (*You may assume that areas of circles are proportional to the squares on their diameters and can use modern algebra/trigonometry if you wish*)

3.4 Archimedes of Syracuse

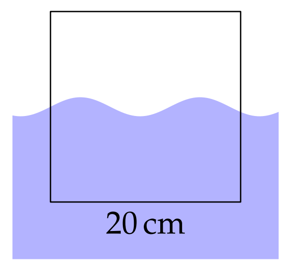
Syracuse is on the island of Sicily at the foot of the Italian peninsula. During Archimedes' lifetime (287–212 BC), Syracuse was a Greek city-state, though its location directly between Carthage (North Africa) and Rome made Sicily a target for both.[20] Archimedes famously helped defend Syracuse against the Romans using catapults, though he ultimately died at their hands as the city fell. He is believed to have travelled to Alexandria in his youth and perhaps studied with scholars at the library, including Eratosthenes (pg. 42).

With repeated demonstrations of both and theoretical and practical genius, Archimedes is a strong candidate for the title of Greatest Mathematician of the Ancient World. Beyond his anti-Roman catapults, he is credited with a large number of inventions and technical innovations, including *Archimedes' screw,* still used in modern irrigation systems to elevate water.

Hydrostatics

Archimedes' principle states that an object immersed in water loses weight equal to that of the displaced water. A famous, but possibly apocryphal, story recounts his discovery of this principle while in the bath (shouting "Eureka!" (I have found it!)); the principle was applied to detect whether a smith had used all the gold he had been given in the manufacture of a crown (it didn't end well for the smith...).

For example, suppose a cube with side length 20 cm floats such that the water-line is halfway up the cube. By Archimedes' principle, the weight of the cube is the same as that of the volume of displaced water, namely

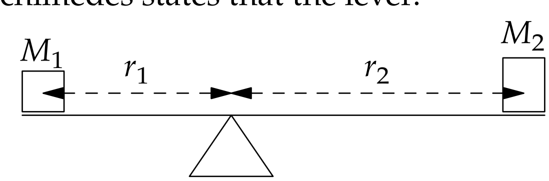


$$20 \times 20 \times 10 = 4000\,\text{cm}^3$$

This has a weight (mass) of roughly 4 kg.

Levers

Archimedes' employed levers for numerous practical purposes and abstract calculations. Given masses M_1, M_2 located distances r_1, r_2 from a pivot, Archimedes states that the lever:



- Balances $\iff M_1 : M_2 = r_2 : r_1$
- Rotates clockwise $\iff M_1 : M_2 < r_2 : r_1$
- Rotates counter-clockwise $\iff M_1 : M_2 > r_2 : r_1$

For example, to find the mass M_2 required to balance a lever given $M_1 = 12\,\text{lb}$, $r_1 = 4\,\text{ft}$ and $r_2 = 3\,\text{ft}$, Archimedes would have observed that

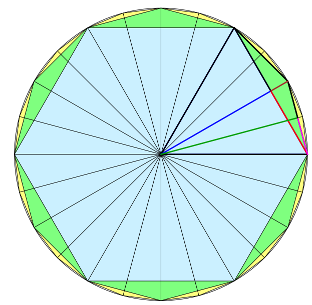
$$M_2 : M_1 = 4 : 3 \implies M_2 = 16\,\text{lb}$$

The modern approach is to compare the *torques* or *moments* $\tau_1 = M_1 r_1$ and $\tau_2 = M_2 r_2$ (the Greeks would not have multiplied non-numerical quantities in this fashion).

[20] During the Second Punic War, the Carthaginian general Hannibal invaded the Italian peninsula (from Iberia/Spain) by famously taking elephants across the Alps. After Hannibal assumed control of most of southern Italy, Syracuse allied itself with Carthage in 215 BC. The Romans besieged Syracuse from 213 and captured it in 212.

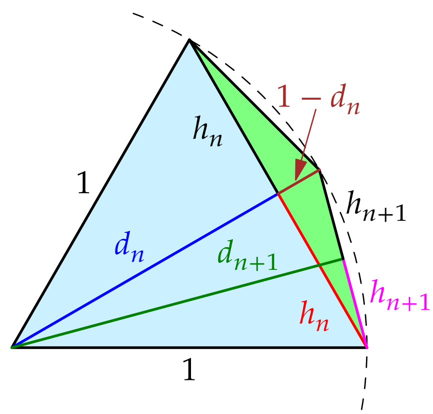
The Method of Exhaustion

Archimedes used limit-like arguments to estimate, and sometimes exactly compute (Exercise 7), various quantities. One famous example comes from his *Measurement of a Circle* (c. 250 BC).[21]



1. Find the perimeter C_0 of an inscribed regular hexagon in a circle of radius 1.
2. Halve the central angle and repeat for a regular dodecagon.
3. Repeat the halving process: Archimedes did this with 24-, 48- and 96-gons to obtain an increasing sequence of perimeters C_n for the $6 \cdot 2^n$-gon, bounded above by that of the circle (2π).
4. Repeat the same calculation with exscribed polygons to obtain a decreasing sequence of over-estimates.
5. Archimedes' 96-gons yielded the estimates $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

Archimedes' angle-halving process relied on a geometric induction step using pure geometry (see, e.g., Exercise 6). Rather than struggle with such language, here is a version using modern algebra: while the notation would be alien to Archimedes, it uses nothing but Pythagoras.



A chord of length $2h_n$ creates an isosceles triangular wedge of the circle with altitude d_n (the initial hexagon A_0 consists of six of these with $2h_0 = 1$). Halve the central angle to create two new isosceles triangles with altitude d_{n+1} and chord $2h_{n+1}$. Apply Pythagoras' three times:

$$1 = d_n^2 + h_n^2 \qquad (2h_{n+1})^2 = h_n^2 + (1 - d_n)^2$$
$$1 = d_{n+1}^2 + h_{n+1}^2$$

Expand and cancel to obtain

$$d_{n+1}^2 = \frac{1}{2}(1 + d_n), \qquad h_{n+1}^2 = 1 - d_{n+1}^2 = \frac{1}{2}(1 - d_n)$$

Since $d_0 = \frac{\sqrt{3}}{2}$ and $h_0 = \frac{1}{2}$, we may compute the entirety of both sequences, for instance:

$$h_1 = \frac{1}{2}\sqrt{2 - \sqrt{3}}, \quad h_2 = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{3}}}, \quad h_3 = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}}, \quad \ldots$$

The circumference of the $6 \cdot 2^n$-sided polygon is $C_n = 12 \cdot 2^n h_n$. The 96-gon required Archimedes to estimate

$$C_4 = 196h_4 = 96\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} > 6.282 \implies \pi > \frac{1}{2}C_4 > 3.141 > 3\frac{10}{71}$$

[21] The real number π (the ratio of the circumference to the diameter of a circle) is sometimes known as *Archimedes' constant*, in part because of these astonishing efforts to, in essence, estimate it. The symbol π didn't acquire this meaning until 1648, and the association only became widespread due to Euler's influence in the 1700s.

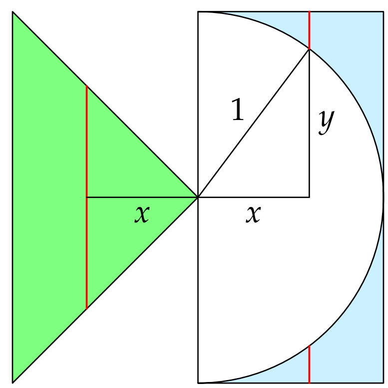
In the same manuscript, Archimedes applies the method of exhaustion to prove that the area of a circle is equal to that of a triangle whose sides are its radius and circumference: in modern language $A_\circ = \frac{1}{2}rc\,(= \frac{1}{2}r \cdot 2\pi r = \pi r^2)$.

Is Archimedes the founder of calculus?

In 1906, a previously unknown work of Archimedes was discovered. In this *Method of Mechanical Theorems* Archimedes explains how to compare the areas and volumes of geometric objects by balancing infinitesimal slices using a lever. These arguments looks remarkably like modern calculus; he could perhaps claim to be its earliest practitioner by 1800 years! The method was outlined in a letter to Eratosthenes (librarian in Alexandria) and includes an argument for proving Archimedes' favorite theorem, a picture of which was engraved on his tomb.

Theorem. *A cone, hemisphere and cylinder with the same base and height have volumes in the ratio* 1 : 2 : 3. *Using modern formulæ, if the height is r, then the volumes are* $\frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3$.

Here is a modernized version illustrating Archimedes' approach. Suppose the base has radius 1, remove the hemisphere from the cylinder and place to the right. Compare the cross-sections the same distance x from the apex of the cone.



- The circular cross-section of the cone has radius x, and area is proportional to the square on the radius: πx^2.
- The cross-section of the right-hand volume is an annulus, whose area is proportional to a difference of squares. By Pythagoras,' its cross-sectional area is

$$\pi(1^2 - y^2) = \pi\big(1 - (1 - x^2)\big) = \pi x^2$$

The cross-sections therefore balance with respect to a horizontal lever with pivot at the center of the figure. Archimedes concludes that the two volumes also balance: that is

$$V_{\text{cone}} = V_{\text{cylinder}} - V_{\text{hemisphere}}$$

Combining this with the fact that $V_{\text{cylinder}} = 3V_{\text{cone}}$ (Exercise 8) gives the desired result.

The Method includes several of these calculus-like discussions. While efficient, Archimedes felt that infinitesimal arguments weren't rigorous and provided alternative arguments elsewhere in his writings. The essential problem is this:

> Can we really say that a volume *equals* its cross-sectional areas? If cross-sections have volume, then summing infinitely many produces an infinite volume; if they have no volume, then we can't sum them to compute a volume!

These are really variations of Zeno's paradoxes (pg. 28) regarding infinitesimals!

Archimedes' infinitesimal arguments would be resurrected in the early 1600s by Cavalieri and Galileo as the development of calculus gathered pace. The same duality of presentation characterised these later arguments: Newton and others found the infinitesimal approach efficient, but felt the need to present alternative arguments or pretend that they weren't really using infinitesimals, lest readers suspect that their results were unjustified trickery.

It is tempting to imagine an alternative history if Archimedes' *Method* had been accepted and preserved as part of the Greek canon. If calculus had been developed 1800 years earlier, how would this have affected technological development? Might the space-race have happened in AD 500…?

Other Highlights of the later Greek Period: 300 BC–AD 500

We'll consider ancient astronomy, including Greek contributions, in the next chapter. Here are a few of the other developments of the late Greek period and some historical context.

- Eratosthenes (276–194 BC) grew up in Cyrene (c. 500 miles west of Alexandria in modern Libya) and moved to Alexandria in adulthood where he eventually became its librarian. Amongst other things, he is credited with a simple algorithm for finding primes: the *Sieve of Eratosthenes*.
  - List the integers $n \geq 2$.
  - Leave 2 and delete all its multiples.
  - Leave 3 and delete its multiples.
  - Repeat ad infinitum: each time one reaches a number, leave it and delete its multiples.
  - The remaining list contains all the primes.
- Apollonius (225 BC) produced an eight-volume book on conic sections building on earlier work of Menaechmusus (350 BC).
- By 146 BC the central part of Greek territory had fallen under Roman rule, though Alexandria remained an important place of scholarship. Educated Greeks still spoke and wrote in Greek rather than (Roman) Latin. For context, Julius Caesar ruled Rome around this time (died 44 BC).
- Hipparchus (140 BC) computed chords (essentially sine tables, although the word was not yet used) for astronomical purposes.
- Heron (AD 75) proved the formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of triangle, where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. This was likely known to Archimedes; Heron's work was a compilation of earlier mathematics.
- Around AD 100 the Neopythagorean's worked in Alexandria, studying music, philosophy, and number, with the intent of reviving the teachings of Pythagoras.

- Ptolemy (AD 150) extended the work of Hipparchus on trigonometry, writing the astronomical masterwork *Almagest*. We shall study this, and Hipparchus, in the next chapter.
- Around AD 400, Theon and Hypatia (of Alexandria) produced a widely-read edition of Euclid's *Elements* and improved the exposition of several earlier mathematical topics.
- In AD 395 the Roman empire split into eastern and western parts centered, respectively, on Rome and Byzantium (Constantinople). The western empire rapidly declined under pressure from corruption and external attacks, collapsing completely by 500.

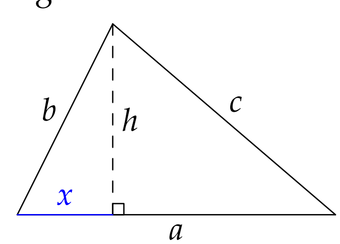
  During this period, Alexandria experienced riots and a bloody power-struggle: Hypatia was murdered by a mob in 415, and the library of Alexandria was severely damaged and possibly destroyed. In 642, Alexandria was captured by the new Islamic caliphate. Much of the material in the library survived by being copied and transported to other houses of learning, particularly Constantinople and Baghdad. For the next 600 years, the knowledge of Alexandria was largely a mystery to (western) Europe.

Exercises 3.4. *Key concepts: Levers, Method of Exhaustion, Early calculus-like methods, Greek knowledge transfer to Byzantium and the Islamic empires*

1. An 8 kg weight is placed 10 m from the pivot of a lever. A 12 kg weight is place 8 m from the pivot on the opposite side of the same lever. Towards which weight will the lever incline? Answer using Archimedes' language.
2. Use Eratosthenes' Sieve (page 42) to find all the primes < 100.
3. (a) Use Heron's formula to compute the area of a triangle with sides 4, 7 and 10.

   (b) We prove Heron's formula using several applications of Pythagoras.'

   Given a triangle with sides a, b, c, let h be the altitude above a, and x the segment pictured (left side of the base). By applying Pythagoras' Theorem to the two right-triangles, show that

   $$x = \frac{a^2 + b^2 - c^2}{2a}$$

   Find h in terms of a, b, c and thus deduce Heron's formula.
4. Use the modern formula $A = \frac{1}{2}ab\sin C$ to prove that, for any $k \in \mathbb{N}$

   $$\frac{1}{2}k\sin\frac{2\pi}{k} < \pi < k\tan\frac{\pi}{k}$$

   In modern language, explain why both sides converge to π.

5. Revisit Archimedes' application of the method of exhaustion to the circumference of a circle (page 40). Suppose C_n is the circumference of a $6 \cdot 2^n$-sided inscribed polygon in a unit circle, A_n its area, and d_n, h_n lengths in that discussion.

   (a) Compute the sequence d_n of altitudes of each isosceles triangle.

   (b) Prove that the area of the n^{th} polygon satisfies the following:

   $$A_{n+1} = \frac{1}{d_n} A_n, \qquad A_n = 6 \cdot 2^n d_n h_n = 3 \cdot 2^n h_{n-1} = \frac{1}{2} C_{n-1}$$

   (*If $C_n \to 2\pi$, it is now immediate that $A_n \to \pi$*)

   (c) Show that the circumference of the corresponding exscribed polygon is $C_n^{\text{ex}} = \frac{1}{d_n} C_n$. Use a calculator to evaluate C_4^{ex} and therefore confirm Archimedes' observation that $\pi < 3\frac{1}{7}$.

6. Instead of modern algebra, Archimedes used several geometric lemmas to help find the areas of polygons inscribed in and circumscribing circles. Here is one; prove it!

   Let $\overline{OA}$ be the radius of a circle and $\overline{AC}$ be tangent to the circle at A. Let D lie on $\overline{AC}$ such that $\overline{OD}$ bisects $\angle COA$. Then

   $$\frac{|DA|}{|OA|} = \frac{|CA|}{|CO| + |OA|} \quad \text{and} \quad |DO|^2 = |OA|^2 + |DA|^2$$

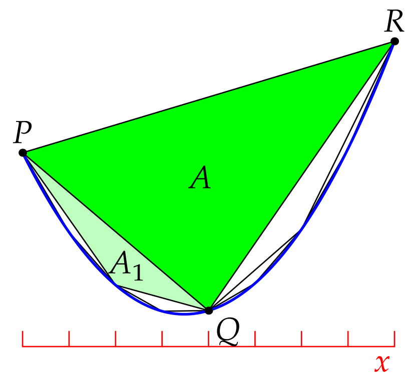
   (*Hint: draw a picture and let T be the intersection of the circle and $\overline{OC}$*)

7. Archimedes evaluated the area inside a parabola using the method of exhaustion in a calculation that amounts to a evaluating a geometric series. Use modern algebra for this question.

   (a) Suppose $y = a+bx+cx^2$ is the equation of a parabola. If P, Q, R have x co-ordinates in an arithmetic sequence $x-\epsilon, x, x+\epsilon$, show that the area of $\triangle PQR$ is $A = |c|\epsilon^3$: importantly, this is *independent of x*.

   (b) With reference to the picture, explain why the areas of the labelled triangles satisfy $A_1 = \frac{1}{8}A$.

   (c) Use a geometric series to prove that the area inside the parabola bounded by $\overline{PR}$ is $\frac{4}{3}A$



8. How might Archimedes have applied the result of Exercise 7 to show that the volume of a cone equals $\frac{1}{3}$ that of a cylinder with the same base and height?

   (*Hint: think about cross-sections of the cone and the cylinder*)