4 Ancient Astronomy & Trigonometry

4.1 Mathematical Modeling of the Sky in Ancient Times

Astronomy is one of the most important drivers of historical mathematical development. Indeed, as we'll see, early *trigonometry* (*triangle-measure*) was primarily developed to facilitate astronomical computations. Measuring the sky has many practical benefits: for instance,

Calendars The phases of the moon (whence *month*), the seasons, and the solar year are paramount. An accurate calendar are essential for food production, gathering and hunting: *When* will the rains come? *When* should we plant/harvest? *When* will the deer return?

Navigation In the northern hemisphere, stars appear to orbit *Polaris* (the pole star), which thus provides a fixed reference point/direction in the night sky. As humans travelled further afield, accurate computations became increasingly important.

However, practicality was likely not the primary reason for the ancient study of the heavens...

Religion and Astrology

In modern times, we distinguish *astronomy* (the science) from *astrology* (how the heavens influence our lives). For most of human history, however, the two were inseparable. In light-polluted modern cities, it is hard to imagine the significance the night sky held for our ancestors; even a couple of centuries ago. Almost all religions imbue the heavens with deep meaning; the development of mathematical tools and technologies with which to model, measure and describe the sky was therefore a religious imperative. Here are a small number of examples of the deep relationship between astronomy, astrology and culture.

- The concept of *heaven* as the domain of the gods transcends many religions, whether explicitly in the sky or simply atop a high mountain (e.g., Olympus in Greek mythology, Moses ascending Mt. Sinai, etc.).
- Many ancient structures were constructed in alignment with heavenly objects:
  - Ancient Egyptians considered the region of the night sky around Polaris as their heaven, and viewed various stars are sacred. Pyramids included shafts emanating from the burial chamber pointing at specific regions of the sky so that the deceased could 'ascend to the stars.'
  - Several Mayan temples are oriented to the rising or setting sun on the solstices (page 45). Similar alignments are found elsewhere in the Americas and worldwide.
  - Venus and Sirius—respectively the brightest planet and star in the night sky—were also important objects of alignment for ancient structures.
- The *star in the east* is associated to the birth of Jesus in Christianity.

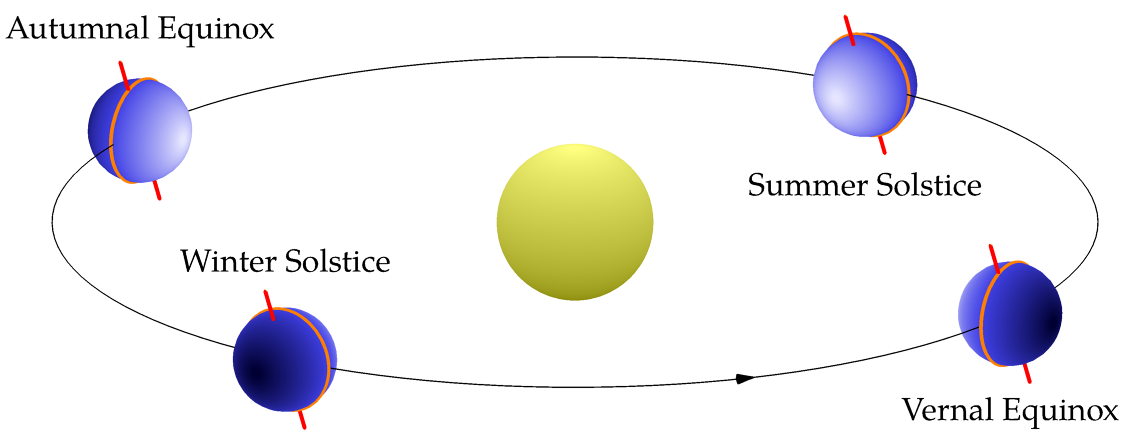
- Muslims orient themselves towards Mecca when at prayer; we'll see later how this direction (the *qibla*) may be computed, though the required data is astronomical. While the same mathematics enables (practical) long-distance navigation, once again we see the religious imperative at work.

- The western zodiac comes from pre-1000 BC Babylon. A Babylonian tablet dating to 686 BC describes 60–70 constellations and stars with aspects familiar to modern astrologers, including Taurus, Leo, Scorpio and Capricorn. During the same period Chinese and Indian astronomers developed different systems of constellations.[22]

- Calendars mark religious festivals, practices and even the age of the world.
  - The traditional Hebrew calendar dates the beginning of the world to 3760 BC.
  - The Mayan long count calendar dates the creation of the world to 3114 BC.
  - The modern Gregorian calendar was created to facilitate the accurate determination of Easter, whose date is defined by astronomical observation (first Sunday after the full moon after the vernal equinox).

Basic Astronomical Terminology: Measuring the Heavens

Seasonal variation exists because Earth's axis is tilted roughly 23.5° with respect to the *ecliptic* (sun-earth orbital plane). Summer is when the axis is tilted towards the sun (in that hemisphere), resulting in more sunlight and longer days. Astronomically, there are four key dates.



The ecliptic, Earth's axis and the day-night boundary.

Summer/Winter Solstices (≈ 21st June/December[23]) At the summer solstice, the north pole is tilted maximally towards the sun; the setting sun is most northerly and (north of the tropics) the noon sun is highest in the sky. The opposite occurs on the Winter solstice.

Solstice comes from Latin meaning *sun stationary*, reflecting the fact that the location of the rising/setting sun and its maximal elevation have their extremes on the solstices.

[22]Chinese astronomy has 28 constellations (or *mansions*). As a point of comparison, Taurus corresponds roughly to the Chinese 'White Tiger of the West' (*Baihu*, and similar terms in other East-Asian languages).

[23]Descriptions are for the northern hemisphere.

Vernal/Autumnal Equinoxes ($\approx 21^{\text{st}}$ March/September) On these dates, the earth's axis is perpendicular to the Sun-Earth orbital radius.

Equinox means *equal night*. At the equinoxes, Earth's axis passes through the day-night boundary: day and night both last approximately 12 hours everywhere.

Dating back to the ancient Babylonians, measurements were conducted relative to this set-up.

Fixed stars These form the background canvas with respect to which everything else is measured. From the Earth, the *ecliptic* is the sun's apparent path over the course of a year viewed against the fixed stars. The movements of planets (*wandering stars*) are also recorded relative to this background.

Celestial longitude Measured from zero to 360° around the ecliptic with 0° fixed by the vernal equinox. One degree corresponds approximately to the sun's apparent daily movement. The ecliptic is divided into twelve equal segments: Aries is 0–30° (March to April 21^{st}); Taurus is 30–60°, etc.

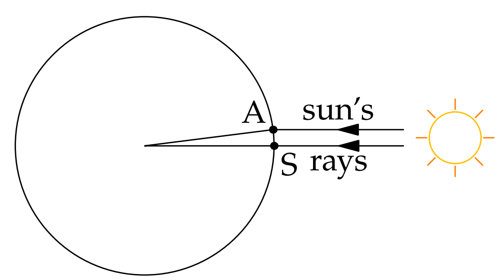
Celestial latitude Measured in degrees north or south of the ecliptic;the sun has latitude zero.

This essentially Babylonian formulation was co-opted by the Greeks, and thence transferred to Islam and medieval Europe. The Greeks kept the Babylonian base-60 degrees-minutes-seconds system, which, with minor modifications, persists to this day.[24]

The Circumference of the Earth

One of the earliest problems in astronomy is to find the size of the earth. Eratosthenes of Cyrene (c. 200 BC, page 42) performed one of the first accurate estimations by measuring the sun's rays at noon in two different places.

- Syene (modern-day Aswan, Egypt) is roughly 5,000 *stadia* south of Alexandria.
- When the sun is directly overhead at Syene, its inclination at Alexandria is roughly $7°12' = \frac{1}{50} \cdot 360°$.
- The circumference of the earth is therefore approximately $50 \cdot 5000 = 250,000$ stadia.



The angle of the sun's rays is trivial to measure: place a stick in the ground and look at its shadow! Eratosthenes' original calculation is lost, though it was a little more complicated than the above. From other sources, historians have inferred that Eratosthenes' *stadion* was ≈ 172 yards, making his approximation for the circumference of the earth $\approx 24,500$ miles, astonishingly accurate in comparison to the modern value of $\approx 25,000$ miles. Later mathematicians provided other estimates based on other locations, but the basic method was the same.

[24] Modern astronomers measure latitude and longitude (declination/right-ascension) with respect to Earth's equatorial plane rather than the ecliptic. Such *equatorial co-ordinates* are first known to have been introduced by Hipparchus of Nicaea (page 51). Right-ascension is measured in hours-minutes-seconds rather than degrees, where 24 hours = 360°, though modern scientific practice is to use decimals rather than sexagesimal minutes and seconds.

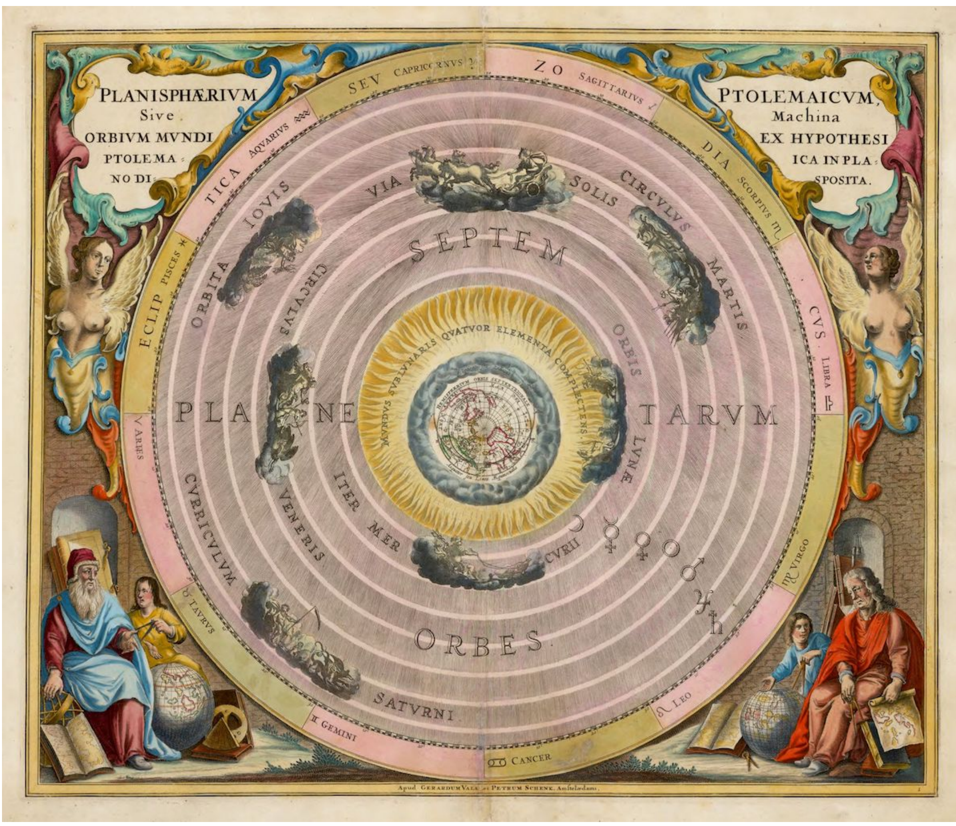
Modelling the Heavens

Early Greek analysis combines two key philosophical ideas.

Geocentrism The earth is stationary and the celestial sphere rotates around it once per day.

Spherical/Circular perfection The ‘perfect design’ of the universe must be based on uniform circular motion.

Spheres and circles were everywhere in Greek models: Earth was (almost) a perfect sphere; planets and the fixed stars were attached to concentric spherical shells centered on the earth. The overall structure is summarized in the picture below.



The Geocentric (Ptolemaic) model: plate from Andreas Cellarius’ 1660 Celestial Atlas

Unfortunately for the Greeks, observations suggest two major contradictions:

Variable brightness Heavenly bodies, particularly planets, have non-constant apparent brightness.

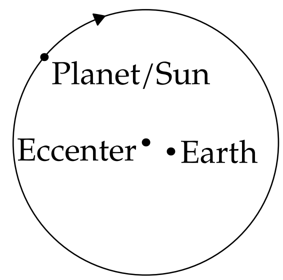
Retrograde motion Planets mostly follow the east-west motion of the heavens, though sometimes they are seen to slow down and reverse course.

If planets move at constant speed around circles/spheres centered on the earth, how can these observations be explained? Attempts to produce accurate models while preserving spherical/circular motion led to the development of new mathematics. Eudoxus of Knidos (c. 370 BC, page

28) made an early attempt to address this via a gimbal-like model where each concentric sphere has its poles attached to the sphere outside it. Retrograde motion can be produced with the model, but not the variable brightness of stars and planets; the model is also highly complex and would have been far beyond ancient Greek abilities to compute with.

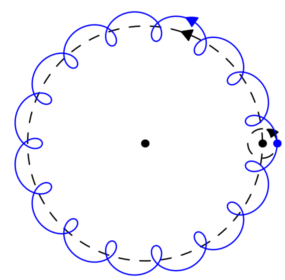
Epicycles & Eccentric Orbits Apollonius of Perga ($2^{nd}/3^{rd}$ C. BC) is most famous mathematically for his study of conic sections, but is relevant here for developing two models of solar/planetary motion.

In the *eccenter* model, a planetary/solar orbit is a circle (the deferent) whose center is *not* the earth. This straightforwardly addresses the problem of variable brightness since the planet is not a fixed distance from the earth.



The obvious question is *why?* What philosophical justification could there be for the eccenter?

Apollonius' second approach used *epicycles*: small circles attached to a larger circle—you'll be familiar with these if you've played with the toy *Spirograph*. An observer at the center sees the apparent brightness change, and potentially observes retrograde motion. In modern language, the motion is parametrized by the vector-valued function

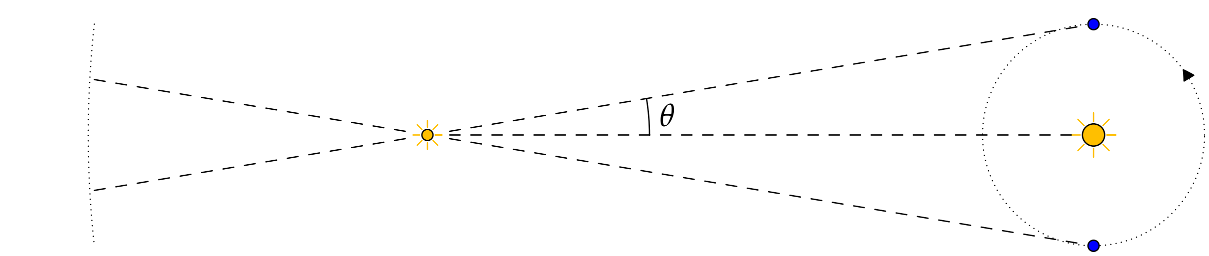


$$\mathbf{x}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} + r\begin{pmatrix}\cos\psi t\\ \sin\psi t\end{pmatrix}$$

where R, r, ω, ψ are the radii and frequencies (rad/s) of the circles.

Combining these models permitted very complex motion, though calculation was difficult, requiring finding lengths of chords of various circles from a given angle, and vice versa. It is from this requirement that some of the earliest notions of trigonometry arose.

Why not Heliocentrism? One might ask why the Greeks didn't make the 'obvious' fix and place the sun at the center of the cosmos. In fact Aristarchus of Samos (c. 310–230 BC) did precisely this, suggesting that the fixed stars were really just other suns at exceptional distance! However, the great thinkers of the time (Plato, Aristotle, etc.) had a very strong objection to Aristarchus' proposal: *parallax*.



If the earth moves round the sun and the fixed stars are really independent objects, then the position of a nearer star should appear to change throughout the year. The angle θ in the picture is the *parallax* of the nearer star. Unfortunately for Aristarchus, the nearest star to our sun exhibits only 0.77 arcseconds ($0.0002°$) of parallax, a quantity far beyond Greek technical ability to measure directly.[25]

It took 2000 years before the work of Copernicus, Kepler & Galileo in the 15-1600s forced astronomers to take *heliocentric* models seriously (*Helios* is the Greek sun-god).

Exercises 4.1. *Key concepts: Reasons for studying the sky, Solstice, Equinox, Ecliptic, Geocentrism, Heliocentrism*

1. *Sirius*, the brightest star in the sky, is 2.64 parsecs (8.6 light-years) from the sun. Use modern trigonometry to find its parallax.

2. The tropic of cancer is the line of latitude (approximately) 23.5° north of the equator marking the locations where the sun is directly overhead at noon on the summer solstice.[26] At the arctic circle on the *winter* solstice, the sun is precisely on the horizon.

   (a) Explain why the latitude of the arctic circle is 66.5° north.

   (b) Find the angle the sun makes *above* the horizon at the arctic circle at noon on the summer solstice.

3. Consider the epicycle model where the position vector of a planet is given by

$$\mathbf{x}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} + r\begin{pmatrix}\cos\psi t\\ \sin\psi t\end{pmatrix}$$

   (a) Suppose $R = 4$ and $r = 1$, $\omega = 1$ and $\psi = 2$, so that the epicycle rotates twice every orbit. Sketch a picture of the full orbit.

   (b) Suppose that ω, ψ are positive constants. Prove that an observer will see retrograde motion if and only if $r\psi > R\omega$.
   (*Hint: differentiate* $\mathbf{x}'(t)$ *and think about its direction*)

[25] The astronomical unit of one *parsec* is the distance to a star exhibiting one arc-second ($\frac{1}{3600}°$) of parallax: roughly 3.3 light-years or 3×10^{13} km, an unimaginable distance to anyone before the scientific revolution. Since the nearest star to our sun, *Proxima Centauri*, lies 4.2 light years = 0.77 parsecs away, the rejection of Aristarchus' hypothesis is entirely understandable!

[26] Syene (page 47) is almost exactly on the Tropic of Cancer.

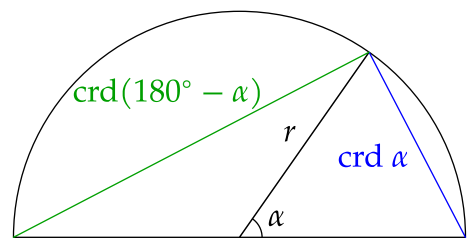
4.2 The Birth of Trigonometry: Hipparchus and Ptolemy

The term *trigonometry* (*triangle measure*) dates to AD 1600, but the key ideas are much older, and originally weren't about triangles at all. In this section we discuss the work of two key figures in the development of early trigonometry.

Hipparchus of Nicaea/Rhodes (c. 190–120 BC)

Born in Nicaea (northern Turkey) but doing much of his work on the Mediterranean island of Rhodes, Hipparchus was one of the pre-eminent Greek astronomers. He applied Apollonius' eccenter and epicycle models to Babylonian lunar eclipse observations. As part of this work, he needed to accurately compute chords of circles. His *chord tables* are acknowledged as the earliest lists of trigonometric values.

In an imitation of Hipparchus' approach, we define a function crd which returns the length of the chord in a given circle subtended by a given angle. In modern language



$$\operatorname{crd}\alpha = 2r\sin\frac{\alpha}{2}$$

Hipparchus chose a circle with circumference 360 (in fact he used $60 \cdot 360 = 21600$), whence $r = \frac{21600}{2\pi} \approx 57,18;$ (base-60). Note that this is sixty times the number of *degrees per radian*.[27] His chord table was constructed starting with two obvious values:

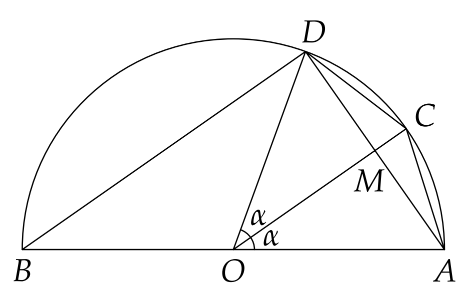
$$\operatorname{crd}60^\circ = r = 57,18; \qquad \operatorname{crd}90^\circ = \sqrt{2}r = 81,2;$$

By Thales' Theorem, the large triangle is right-angled. Pythagoras' now computes the chord of the supplementary angles

$$\operatorname{crd}(180^\circ - \alpha) = \sqrt{(2r)^2 - (\operatorname{crd}\alpha)^2} = 2r\sqrt{1-\sin^2(\alpha/2)} = 2r\cos\frac{\alpha}{2}$$

Pythagoras' was used to halve and double angles analogously to Archimedes' measurement of the circle (page 40). We rewrite the argument in this language.

In the picture, we double the angle α. Plainly M is the midpoint of $\overline{AD}$ and $|DB| = \operatorname{crd}(180^\circ - 2\alpha)$. Since $\angle BDA = 90^\circ$, it follows that $\overline{BD}$ is parallel to $\overline{OM}$ and so



$$|OM| = \frac{1}{2}|BD| = \frac{1}{2}\operatorname{crd}(180^\circ - 2\alpha)$$

Now apply Pythagoras to $\triangle CMD$:

[27] One radian is the angle subtended by an arc whose length equals the radius of the circle. Hipparchus does this in reverse: the circumference is fixed so that *degree* measures both subtended angle *and* circumferential distance.

$$
\begin{aligned}
(\operatorname{crd}\alpha)^2 &= \left(\frac{1}{2}\operatorname{crd}2\alpha\right)^2 + \left(r - \frac{1}{2}\operatorname{crd}(180^\circ - 2\alpha)\right)^2 \qquad (|CD|^2 = |DM|^2 + |CM|^2)\\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180^\circ - 2\alpha) + \frac{1}{4}\operatorname{crd}(180^\circ - 2\alpha)^2\\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180^\circ - 2\alpha) + \frac{1}{4}(4r^2 - (\operatorname{crd}2\alpha)^2)\\
&= 2r^2 - r\operatorname{crd}(180^\circ - 2\alpha)\\
&= 2r^2 - r\sqrt{4r^2 - (\operatorname{crd}2\alpha)^2}
\end{aligned}
$$

In modern notation this is one of the double-angle trigonometric identities!

$$4r^2\sin^2\frac{\alpha}{2} = 2r^2 - 2r^2\cos\alpha \iff \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

Example To calculate $\operatorname{crd}30^\circ$, we start with $\operatorname{crd}60^\circ = r$. Then

$$
\begin{aligned}
&\operatorname{crd}120^\circ = \sqrt{4r^2 - r^2} = \sqrt{3}\,r\\
\implies\ &\operatorname{crd}30^\circ = \sqrt{2r^2 - r\operatorname{crd}(180^\circ - 60^\circ)} = \sqrt{2r^2 - \sqrt{3}r^2}\\
&\qquad\quad = \sqrt{2 - \sqrt{3}}\,r
\end{aligned}
$$

In modern language this yields an exact value for $\sin 15^\circ$:

$$\operatorname{crd}30^\circ = 2r\sin 15^\circ \implies \sin 15^\circ = \frac{1}{2}\sqrt{2 - \sqrt{3}}$$

Continuing this process, we obtain $\operatorname{crd}150^\circ = \sqrt{2 + \sqrt{3}}\,r$, whence

$$
\begin{aligned}
&(\operatorname{crd}15^\circ)^2 = 2r^2 - r\operatorname{crd}150^\circ = \left(2 - \sqrt{2 + \sqrt{3}}\right)r^2\\
\implies\ &\operatorname{crd}15^\circ = \sqrt{2 - \sqrt{2 + \sqrt{3}}}
\end{aligned}
$$

Again translating to modern language, this says $\sin 7.5^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{3}}}$.

In similar fashion, Hipparchus computed the chords of 7.5°, 15°, ..., 172.5°, in steps of 7.5°. Of course everything was an estimate since he had to rely on repeated approximations for square-roots. All Hipparchus' original work is lost, so we know of his approach only by reference from others. In particular, while the above method is probably due to Hipparchus, we see it first in the work of Ptolemy, whom we consider next.

Claudius Ptolemy (c. AD 100–170)

Born in Egypt and living much of his life in Alexandria, Ptolemy was a Greek/Egyptian/Roman[28] astronomer and mathematician. Around AD 150, he produced the *Mathematica Syntaxis,* better known as the *Almagest*. This latter term is derived from the Arabic *al-mageisti* (great work), reflecting its importance to later Islamic learning.

The *Almagest* is essentially a textbook covering geocentric cosmology. It shows how to compute the motions of the moon, sun and planets, describe lunar parallax, eclipses, the constellations, and elementary spherical trigonometry (this last is likely courtesy of Menelaus c. AD 100). It contains our best evidence as to the accomplishments of Hipparchus and describes his calculations. The *Almagest* formed the basis of Western & Islamic astronomical theory well into the 1600s.

Ptolemy's Calculations Ptolemy used several innovations to compute more chords at greater accuracy than Hipparchus.

Initial Data Ptolemy took $r = 60$ so that $\operatorname{crd} 60° = 60$. He also had more initial data:

$$\operatorname{crd} 90° = 60\sqrt{2}, \quad \operatorname{crd} 36° = 30(\sqrt{5} - 1), \quad \operatorname{crd} 72° = 30\sqrt{10 - 2\sqrt{5}}$$

Halving/Doubling Angles Ptolemy used what was probably Hipparchus' method:

$$\begin{aligned} \operatorname{crd}^2 \alpha &= 2r^2 - r \operatorname{crd}(180° - 2\alpha) = 60\big(120 - \operatorname{crd}(180° - 2\alpha)\big) \\ \operatorname{crd}(180° - \alpha) &= \sqrt{(2r)^2 - \operatorname{crd}^2 \alpha} = \sqrt{120^2 - \operatorname{crd}^2 \alpha} \end{aligned}$$

approximating square-roots to the desired accuracy. For example,

$$\begin{aligned} \operatorname{crd} 30° &= \sqrt{60(120 - \operatorname{crd} 120°)} = \sqrt{60\left(120 - 60\sqrt{3}\right)} \\ &= 60\sqrt{2 - \sqrt{3}} \approx 31;3,30 \end{aligned}$$

Multiple-Angle Formula Ptolemy computed $\operatorname{crd} 12° = \operatorname{crd}(72° - 60°)$, then halved this for angles of 6°, 3°, 1.5°, and 0.75°. Chords for all integer multiples of 1.5° were computed using multiple-angle/addition formulæ.

Interpolation The observation $\alpha < \beta \implies \frac{\operatorname{crd} \beta}{\operatorname{crd} \alpha} < \frac{\beta}{\alpha}$ (known as Aristarchus' inequality) allowed Ptolemy to compute chords for every half-degree to the incredible accuracy of two sexagesimal places. For approximating between half-degrees, his table indicated how much should be added for each arc-minute $(\frac{1}{60}°)$. To obtain these arc-minute approximations, it is believed Ptolemy computed half-angle chords to an accuracy of *five sexagesimal places* (1 part in over 750 million!). The construction of the chord-table must have been a gargantuan task, one for which Ptolemy likely had much assistance.

[28] The name Ptolemy (Ptolemaeus) is Greek, while Claudius is Roman, reflecting the changing culture in Egypt.

Example Part of Ptolemy's chord table is shown below. Ptolemy used ionic Greek enumeration in sexagesmial, and used the special symbol $\angle'$ for $\frac{1}{2}$. The third line of the table reads

$\alpha\angle'$	$\alpha\ \gamma\beta\ \iota\epsilon$	$o\ \alpha\ \beta\ \nu$
$1\frac{1}{2}^\circ$	$1;34,15$	$;1,2,50$

The first two columns state that crd 1.5° = crd $1^\circ 30' = 1;34,15$ to two sexagesimal places. This is the approximation

$$1 + \frac{34}{60} + \frac{15}{60^2} = 1.57083\ldots \approx 120 \sin \tfrac{3}{4}^\circ = 1.57075\ldots$$

a phenomenal level of accuracy. The third entry[29] says that we add $;1,2,50$ for every arc-minute beyond 1.5°: for example

$$\text{crd}\, 1^\circ 35' \approx 1;34,15 + 5(;1,2,50) = 1;39,29,10 \approx 1;39,29$$


48 ΚΛΑΥΔΙΟΥ ΠΤΟΛΕΜΑΙΟΥ

ια'. Κανόνιον τῶν ἐν κύκλῳ εὐθειῶν.

περιφε-ρειῶν	εὐθειῶν			ἑξηκοστῶν			
L'	ο	λα	κε	ο	α	β	ν
α	α	β	ν	ο	α	β	ν
αL'	α	λδ	ιε	ο	α	β	ν
β	β	ε	μ	ο	α	β	ν
βL'	β	λζ	δ	ο	α	β	μη
γ	γ	η	κη	ο	α	β	μη
γL'	γ	λθ	νβ	ο	α	β	μη
δ	δ	ια	ιϛ	ο	α	β	μζ
δL'	δ	μβ	μ	ο	α	β	μζ

[29]When writing in sexagesimal, omicron (o)—meaning 70 in ionic enumeration—was freed up. It is used here to indicate the absence of a quantity. Other copies of Ptolemy used a blank space, or a dash —. While omicron is here fulfilling one function of the modern mathematical zero, it wasn't considered a number and elsewhere still meant 70.

Why would Ptolemy have known exact values for $\operatorname{crd} 36°$ **and** $\operatorname{crd} 72°$**?** Everything is in Euclid's *Elements*!

Theorem (Book XIII, Thms. 9 & 10).

1. *In a circle, the sides of a regular inscribed hexagon and decagon are in the golden ratio.*

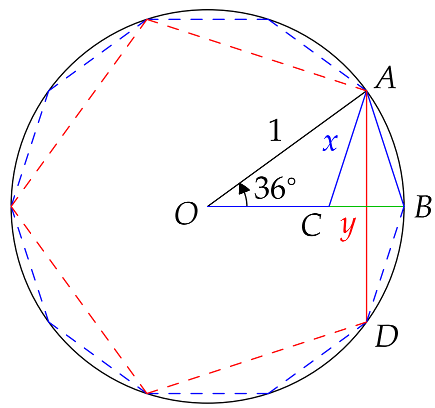
   Since Ptolemy uses a circle of radius 60, for him this ratio is $60 : \operatorname{crd} 36° = 60 : 30(\sqrt{5} - 1)$.

2. *(Thm XIII. 10) In a circle, the square on an inscribed pentagon equals the sum of the squares on an inscribed hexagon and decagon.*

Purely Euclidean proofs are too difficult for us, so here is a way to see things in modern notation.

1. Let $\overline{AB} = x$ be the side of a regular decagon inscribed in a unit circle with center O.

   $\triangle OAB$ is isosceles with angles $36°, 72°, 72°$.
   Let C lie on $\overline{OB}$ such that $\overline{AC} = x$.
   Count angles to see that $\triangle OAB$ and $\triangle ABC$ are similar, that $\angle OAC = 36°$ and so $\overline{OC} = x$.
   Similarity now tells us that



$$x = \frac{1-x}{x} \implies x = \frac{\sqrt{5}-1}{2}$$

   In a circle of radius 60, this gives the exact value

$$\operatorname{crd} 36° = 60x = 30(\sqrt{5} - 1)$$

2. Now let $\overline{AD} = y$ be the side of a regular pentagon inscribed in the same circle. Applying Pythagoras, we see that

$$\left(\frac{y}{2}\right)^2 + \left(\frac{1-x}{2}\right)^2 = x^2$$

   Since $x^2 = 1 - x$, this multiplies out to give Euclid's result

$$y^2 = 1^2 + x^2$$

   from which we obtain the exact value

$$\operatorname{crd} 72° = 60y = 30\sqrt{10 - 2\sqrt{5}}$$

While these values were geometrically precise, Ptolemy used sexagesimal approximations to square-roots to obtain the values stated in his tables:

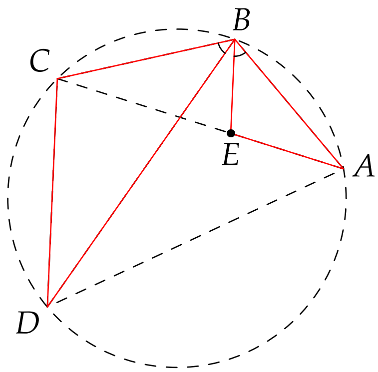
$$\operatorname{crd} 36° = 37;4,55 \qquad \operatorname{crd} 72° = 70;32,3$$

He must have used a far higher degree of accuracy in order to obtain similarly accurate values for other chords.

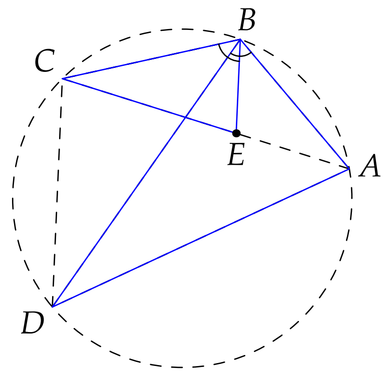
Angle-addition & the Multiple-angle Formula Computation of $\operatorname{crd}(\alpha \pm \beta)$ was facilitated by versions of the familiar multiple-angle formulæ.

Theorem (Ptolemy's Theorem). *Suppose a quadrilateral is inscribed in a circle. Then the product of the diagonals equals the sum of the products of the opposite sides.*[30]

Proof. Choose E on $\overline{AC}$ such that $\angle ABE \cong \angle DBC$. Then $\angle ABD \cong \angle EBC$. Since $\angle BAE \cong \angle BDC$ are inscribed angles of the same arc $\overline{BC}$, we obtain two pairs of similar triangles:



$\triangle ABE \sim \triangle DBC$



$\triangle ABD \sim \triangle EBC$

The proof follows immediately: since $\frac{|AE|}{|CD|} = \frac{|AB|}{|BD|}$ and $\frac{|CE|}{|AD|} = \frac{|BC|}{|BD|}$, we have

$$|AC|\,|BD| = (|AE| + |CE|)\,|BD| = |AB|\,|CD| + |AD|\,|BC|$$

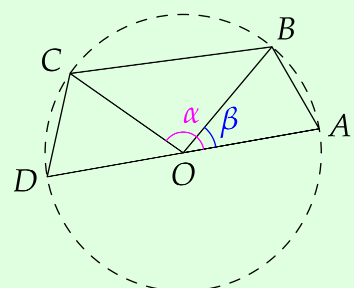
∎

Corollary. *If $\alpha > \beta$, then*

$$120 \operatorname{crd}(\alpha - \beta) = \operatorname{crd}\alpha \operatorname{crd}(180^\circ - \beta) - \operatorname{crd}\beta \operatorname{crd}(180^\circ - \alpha)$$

In modern language, divide through by 120^2 to obtain

$$\sin\frac{\alpha - \beta}{2} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\alpha}{2}$$



Proof. If $|AD| = 120$ is a diameter of the pictured circle, then Ptolemy's Theorem says

$$\operatorname{crd}\alpha \operatorname{crd}(180^\circ - \beta) = \operatorname{crd}\beta \operatorname{crd}(180^\circ - \alpha) + 120 \operatorname{crd}(\alpha - \beta)$$

∎

Similar expressions for $\operatorname{crd}(\alpha + \beta)$ and $\operatorname{crd}(180^\circ - (\alpha \pm \beta))$ were also obtained, essentially recovering all versions of the modern multiple-angle formulæ for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$.

[30] It is generally considered that this result predates Ptolemy, though there is some debate as to whether it belongs in the *Elements*. Book VI traditionally contains 33 propositions, however some editions append four corollaries, of which Ptolemy's Theorem is the last (Thm VI. D).

Example 1 Here is how Ptolemy would have calculated crd 42°.
Let $\alpha = 72°$ and $\beta = 30°$, then

$$120 \operatorname{crd} 42° = \operatorname{crd} 72° \operatorname{crd} 150° - \operatorname{crd} 30° \operatorname{crd} 108°$$

Since $\operatorname{crd} 72° = 30\sqrt{10 - 2\sqrt{5}}$ is known, so also is

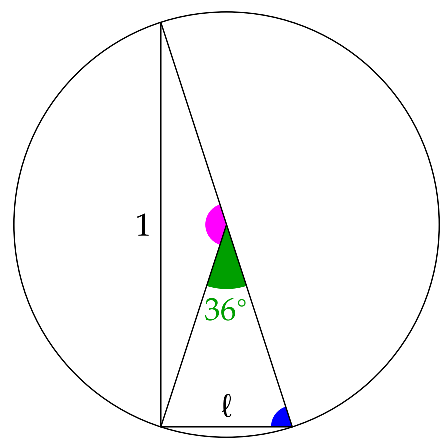
$$\operatorname{crd} 108° = \operatorname{crd}(180° - 2 \cdot 36°) = 120 - \frac{1}{60} \operatorname{crd}^2 36° = 30(1 + \sqrt{5})$$

The multiple angle formula then shows that

$$\begin{aligned}\operatorname{crd} 42° &= \frac{1}{120}\left(30\sqrt{10 - 2\sqrt{5}} \cdot 60\sqrt{2 + \sqrt{3}} - 60\sqrt{2 - \sqrt{3}} \cdot 30(1 + \sqrt{5})\right) \\ &= 15\left(\sqrt{10 - 2\sqrt{5}} \cdot \sqrt{2 + \sqrt{3}} - (1 + \sqrt{5})\sqrt{2 - \sqrt{3}}\right) \\ &\approx 43;0,15 \approx 43.0042\end{aligned}$$

Note all the square-roots which had to be approximated!

Example 2 The *Almagest* also contained many practical examples. Here is one such.
A stick of length 1 is placed in the ground. The angle of elevation of the sun is 72°. What is the length of its shadow?
Ptolemy instructs us to draw the circumcircle of the triangle created by the stick and its shadow.
The lower isosceles triangle has base angles 72° and the length of the shadow is ℓ. The ratio of the chords is then computed:



$$1 : \ell = \operatorname{crd} 144° : \operatorname{crd} 36°$$

$$\Longrightarrow \ell = \frac{\operatorname{crd} 36°}{\operatorname{crd} 144°} = \frac{30(\sqrt{5} - 1)}{30\sqrt{10 + 2\sqrt{5}}} \approx 0.32491$$

This is precisely cot 72°, though Ptolemy had no such notion.

Exercises 4.2. *Key concepts: Early trigonometry is about* ***chords*** *of circles, Half-angle formulae using Pythagoras', Multiple-angle formulae, Ptolemy's chord table*

1. What are the exact values of sin 36° and sin 18°?
2. Calculate crd 150°, crd 165°, and $\operatorname{crd} 172\frac{1}{2}°$ using the method of Hipparchus.
   (*Leave your answers as a multiple of* $r = \operatorname{crd} 60°$)
3. Find crd 56′ (arc-minutes!) to two sexagesimal places using Ptolemy's table of chords. Compare your answer (using decimals) to the exact value 120 sin 28′.
4. Find the exact value of crd 54°

5. Restate Ptolemy's interpolation formula $\alpha < \beta \implies \frac{\operatorname{crd}\beta}{\operatorname{crd}\alpha} < \frac{\beta}{\alpha}$ in terms of the sine function. What facts about $\frac{\sin x}{x}$ does this reflect? Can you prove it?

6. Prove the following using Ptolemy's Theorem.

$$120 \operatorname{crd}\big(180° - (\alpha + \beta)\big) = \operatorname{crd}(180° - \alpha)\operatorname{crd}(180° - \beta) - \operatorname{crd}\alpha \operatorname{crd}\beta$$

To what does this correspond in modern language?

7. Use Ptolemy's Theorem to establish a version of the double-angle formula:

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

(*Hint: draw a symmetric quadrilateral one of whose diagonals is a diameter*)

8. Calculate the length of a noon shadow of a pole of length 60 using Ptolemy's methods:
   (a) On the vernal equinox at latitude 40°.
   (b) At latitude 36° north on both the summer and winter solstices.
   (*Hint: recall Exercise 4.1.2*)