6 Indian and Islamic Mathematics

6.1 India, the Hindu–Arabic Numerals & Zero

The Indian/South Asian subcontinent is bordered to the north by the Himalayan mountains and to the east by dense jungle. Its primary historical frontier comprised the fertile Indus valley to the west—now the central corridor of Pakistan—where recorded civilization dates to at least 2500 BC. During the first millennium BC, Hinduism developed as an amalgamation of previous practices and beliefs; Buddhism and Jainism began to spread later in this period, particularly in the Ganges valley further east.

Alexander the Great's conquests reached the Indus in 326 BC, bringing Greek, Babylonian and Egyptian knowledge in his wake. The Greek overlords he left behind were rapidly overthrown and the subcontinent became largely unified under the Mauryan Empire for the next 150 years. After this came 1000 years of shifting control with several Persian invasions from the west. Islam came to the Indus around AD 1000; most of India was part of the Islamic Mughal Empire by the 1500s. After the Mughal decline and fragmentation, the British became dominant in 1857.

The modern political situation reflects this complicated history. India gained its independence from Britain in 1947 and was shortly thereafter partitioned according to religion: the greater Indus valley and the lower Ganges/Brahmaputra comprise the modern Islamic states of Pakistan and Bangladesh, with the majority of the landmass becoming the nominally secular but majority Hindu nation state of India. The upper Indus valley (Kashmir) remains contested and has been the site of several military conflicts between India, Pakistan and China.

Ancient India's contributions to world knowledge and development are significant; India is estimated to have accounted for 25–30% of the world's economy during the 1st millenium AD! It was moreover a technological and cultural crossroads between East (China) and West (Greece, Persia, Rome, etc.): while some trade and knowledge passed north of the Himalayas directly between China and the Middle East/Europe, more percolated slowly through India, being improved upon and given back in turn.

Brahmi Numerals & Numerical Naming

Ancient India is the source of perhaps the most important practical mathematical development in history: the decimal positional system of enumeration, complete with fully-functional zero. The Brahmi numerals, picture below, are one of the earliest antecedents of modern numerals, first appearing around the 3rd century BC.

1 2 3 4 5 6 7 8 9 10

The example dates from around 100 BC and was used in Mumbai/Bombay. Additional symbols denoted multiples of 10, 100, 1000, 10000, etc. The Brahmi system was partly positional, in a manner similar to Chinese enumeration: there was no symbol or placeholder for zero and, for instance, 800 would be written by prefixing the symbol for 100 by that for 8.

Symbols are only part of the story. The modern approach to naming and constructing large numbers can also be linked to the same period. The table below lists names in ancient Sanskrit.

1	2	3	4	5	6	7	8	9
eka	dvi	tri	catur	pancha	sat	sapta	asta	nava
10	20	30	40	50	60	70	80	90
dasa	vimsati	trimsati	catvarimsat	panchasat	sasti	saptati	asiti	navati
100	1000	10000	100000	1000000	10^7	10^8	10^9	10^{10}
sata	sahasra	ayuta	niyuta	prayuta	arbuda	nyarbuda	samudra	madhya

Some of these names no doubt seem familiar since many European languages have Sanskrit roots, e.g., *dva* (Russian *two*), *quatre* (French *four*). Unlike the ancient Chinese, but exactly like modern European languages, there are distinct words for 10 through 90 (instead of, say, *four-ten* for *forty*). The construction of larger numbers is also familiar: for instance, *tri sahasra sat sata panchasat nava* is precisely how we read 3659.

Of course, old languages don't map perfectly onto modern English. For instance, ancient Sanskrit had distinct words for powers of 10 up to 10^{62} (at least!), and employed a version of pre-subtraction: *ekanna-niyuta* meant 'one less than 100000,' or 99999.

Gwalior Numerals

During the first few centuries AD, a fully positional decimal place syste, with zero, came into being. The earliest evidence comes from a manuscript found in Bakhshālī (Pakistan) in 1881, which has been carbon-dated to the 3^{rd} or 4^{th} century; it contains the earliest known version of the modern symbol for zero, a circular dot. It is conjectured that this system was inspired by the Chinese counting-board method, though convincing proof has yet to be uncovered. Regardless of attribution, Chinese mathematicians were copying the method by the 8^{th} century.

The examples below are better understood than the Bakhshālī manuscript and come from Gwalior (northern India) around 876.

0 1 2 3 4 5 6 7 8 9 10

The relation to modern numerals is clear: 0, 1, 2, 3, 7, 9 & 10 are very similiar. Zero has evolved from the Bakhshālī dot to a hollow circle. The symbols for 2 and 3 are conjectured to have developed in an attempt to write earlier versions (the Brahmi numerals) cursively; try writing three horizontal strokes quickly...

The system is fully positional. Below are the numbers 270 and 30984:

Sanskrit is written left-to-right, with the leftmost digits representing the largest powers of 10. Note how zero is used as a placeholder so that, say, 27, 207, and 270 are clearly distinguishable.

Zero: Etymology and Meaning

On the right is a table of modern Sanskrit names and numerals; the digits and names are certainly similar to their Gwalior counterparts.

०	१	२	३	४
0	1	2	3	4
shuunyá	ekaḥ	dvau	tryaḥ	catvāraḥ
५	६	७	८	९
5	6	7	8	9
pañca	ṣaṭ	sapta	aṣṭa	nava

The Sanskrit *shuunyá* means *void* or *emptiness.* It is related to *svi* (hollow), which in turn derives from an ancient word meaning *to grow.* This reflects a major idea within religions of the area, that the void is a source of growth and creativity. Contemplation of the void (the doctrine of Shuunyata) is recommended before composing music, creating art, etc. This contrasts with the Abrahamic religions where the void is something to be feared; an early conception of hell was the eternal absence of God.

The Gwalior numerals travelled westwards. Europe eventually inherited the system via Islam; as such they are today known the *Hindu–Arabic* numerals. Here is a short version of the etymological journey of zero into European languages.

- Arabic had the word *sifr* for empty, so this was used to translate *shuunya*. The double-meaning thus persisted: *al-sifr* was the number zero, whereas *safira* meant *it was empty.*
- Zero came to Europe around 1200, courtesy of Fibonacci. Transliterating from Arabic was inconsistent, resulting in various spellings: *cifra*, *zephyrum* and *zefiro*. The last two of these were already terms for the *west wind* (*zephyr*).
- From *cifra* descend *cipher* (English), *chiffre* (French) and *ziffer* (German): a figure, digit, or code.
- *Zephyrum* and *zefiro* became *zero* in modern Italian, French and then English.

Zero and the Hindu–Arabic numerals also travelled eastwards, with Qin Jiushao introducing the zero symbol into China in the 13$^{\text{th}}$ century.

What is Zero? Our modern understanding of zero is a fusion of several concepts.

Numerical positioning For instance, to distinguish 101 from 11.

Absence of a quantity 101 contains no 10s.

Symbol First a dot (*bindu*), then a circle (*chidra*/*randhra* meaning *hole*). The relationship between *shuunya* and a dot-symbol was established by AD 2-300, as this quote from AD 400 (Vasavadatta) illustrates:

> The stars shone forth, like zero dots [shunya-bindu] scattered as if on a blue rug.
> The Creator reckoned the total with a bit of the moon for chalk.

Mathematical operations By the time of Brahmagupta (7th C.), Indian mathematical texts would contain a section called *shuunya-gania,* describing computations involving zero: addition, multiplication, subtraction, effects on ±-signs, division and the relationship with ∞ (*ananta*). In the 12th century, Bhaskaracharya stated:

> If you were to divide by zero you would get a number that was "as infinite as the god Vishnu."

Other ancient cultures had one or more of these aspects of zero, but the Indians were the first to put them all together.

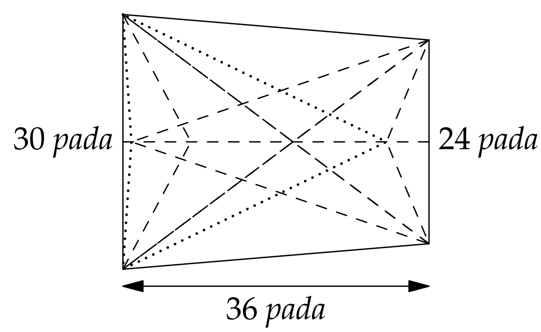
- The Egyptian hieroglyph *nfr* (beautiful/complete) indicated zero remainder in calculations as early as 1700 BC and was also used as a reference point/level in buildings.
- Very late in Babylonian times, a placeholder symbol was used to separate powers of 60. It was not used as a number.
- With the Chinese counting board, an empty space served as a placeholder.
- Various Mesoamerican cultures (e.g., the Maya) used a special symbol as a placeholder.

'Real' Indian Mathematics

Indian mathematicians made great progress on several fronts, not merely with the decimal place system.

Much ancient work was influenced by religion. For instance, the pre-Hindu *sulbasutras* contained instructions for laying out altars using ruler-and-compass constructions. These could be complex, as the base of the *Mahavedi* (great altar) shows: the center line is divided left-to-right in the ratio

1 : 7 : 12 : 11 : 5



and the altar contains five distinct Pythagorean triples!

Of particular importance to our continuing narrative is Indian work on trigonometry. Here are some highlights:

- The early 5th C. text *Paitāmahasiddhānta* seems to be an extension of Hipparchus' work, since it contains a table of chords based on a circle of radius 57,18; rather than Ptolemy's 60.
- Indian mathematicians instituted the use of *half-chords,* in line with our modern understanding of sine. Indeed the word *sine* is the result of a long sequence of (mis)translations and transliterations via Arabic and Latin from the Sanskrit *jyā-ardha* (*chord-half*).[36] The Indians also began to distinguish 'base sine' and 'perpendicular sine' (cosine).

[36]This is also the root of the word *sinus* meaning *bay* or *gulf* (e.g., in your nose).

- Ancient tables of sines/half-chords ranged from 0° to 90° in steps of $3\frac{3}{4}°$, and used linear interpolation to approximate values in between. By 650, Bhramagupta had better approximations and was using quadratic polynomials to interpolate. By 1530, Indian mathematicians had discovered cubic and higher approximations (essentially Taylor polynomials 130 years before Newton) for even greater accuracy of sine, cosine and arctangent.

Navigation was one of the drivers of this development. While Mediterranean sailors rarely strayed long out of sight of land, the Indians sailed the ocean and required accurate measurements to find their latitude.

Exercises 6.1. *Key concepts: Ancient India as a crossroads, Origin of Hindu–Arabic numerals, Zero (meaning and etymology), Contributions to trigonometry*

1. The *Mahavedi* (pg. 71) contains five Pythagorean triples. Find them.

2. To simplify square root expressions, Bhaskara used the formula

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{1}{2}\left(a+\sqrt{a^2-b}\right)} + \sqrt{\frac{1}{2}\left(a-\sqrt{a^2-b}\right)}$$

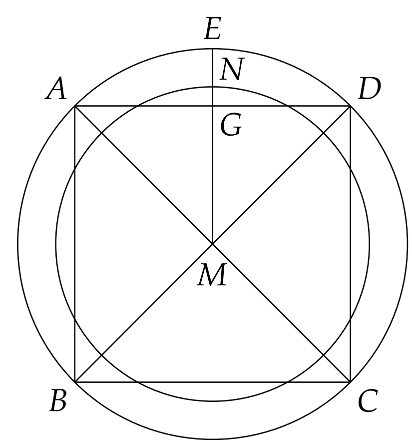
   Prove Bhaskara's formula and use it to simplify $\sqrt{2+\sqrt{3}}$.

3. Here is an Indian method for 'finding' a circle whose area is equal to a given square.

   Given a square $\square ABCD$, let M be the intersection of the diagonals. Draw the circle with center M and radius $\overline{MA}$. Let $\overline{ME}$ be the radius of the circle perpendicular to the side $\overline{AD}$ and let G be where this radius cuts $\overline{AD}$. Let $\overline{GN} = \frac{1}{3}\overline{GE}$. Then $\overline{MN}$ is the radius of the desired circle. Show that if $|AB| = s$ and $|MN| = r$, then

$$\frac{r}{s} = \frac{2+\sqrt{2}}{6}$$

   Show that this implies $\pi \approx 3.08831\ldots$.



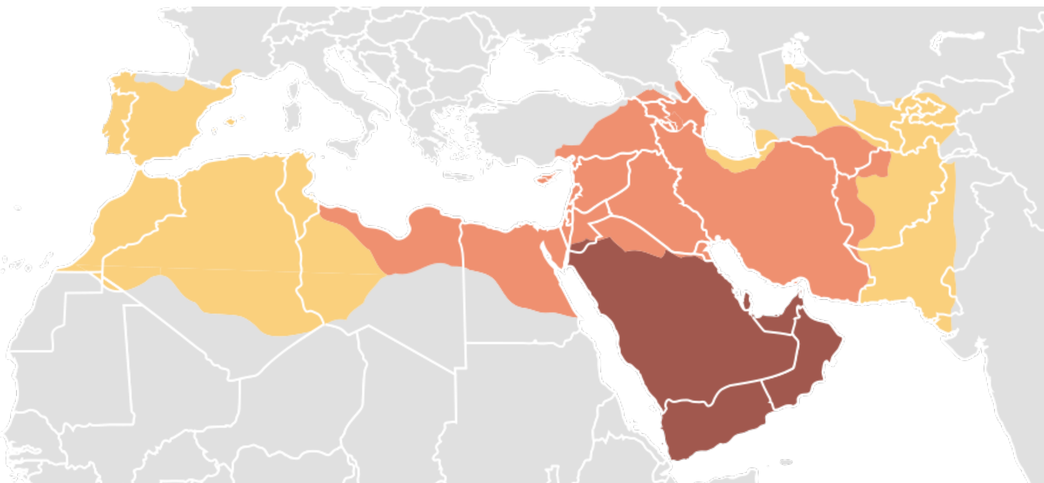
4. Solve the following problem of Mahāvīra.

   > Of a collection of mango fruits, the king took 1/6; the queen took 1/5 of the remainder, and the three chief princes took 1/4, 1/3, 1/2 of what remained at each step. The youngest child took the remaining three mangoes. O you, who are clever in working miscellaneous problems on fractions, give out the measure of that collection of mangoes.

6.2 Islamic Mathematics I: Algebra

Muhammad ibn Abdullah was born in Mecca (modern Saudi Arabia) in 570. Around 610 he began preaching *Islam* (*submission to the will of God*)—the third (chronologically) of the major Abrahamic religions, following Judaism and Christianity. After several years in exile, he returned with an army, conquering Mecca a few years before his death in 632.

Through military conquest, Muhammad's successors rapidly expanded the caliphate (empire). At the time of his death, the Arabian peninsula was Islamic. By 660 Islam had reached Libya and most of Persia, and by 750 extended from Iberia & Morocco to Afghanistan & Pakistan. Serious schisms arose[37] and several successor empires emerged, of which the longest-lasting in the west was the Ottoman Empire (c. 1300–1922). Even though centralized political control ended long ago, Islam remains dominant in the region pictured (with the notable exceptions of Spain and Portugal) and over a greater region of Africa and south-east Asia. [38]



As with the Romans, early Muslims permitted conquered peoples—including Jews and Christians (*people of the book*)—to maintain much of their culture, provided they acknowledged their overlords and paid taxes. Those who converted to Islam could be welcomed as full citizens, though deconversion (apostasy) was not tolerated. Many of the great Islamic thinkers were indeed born on the periphery and traveled to the great centers of learning, particularly Baghdad during the Islamic golden age (8^{th}–13^{th} centuries).

Knowledge was also absorbed from Alexandria and western India (Pakistan). In the mid-700s paper-making came from China, greatly facilitating the dissemination and consolidation of knowledge. Schools (*madrassas*) reflected a strong cultural and religious focus on learning.

The Islamic golden age largely overlapped the European *dark ages* (c. 500–1200) following the fall of Rome, during which European philosophical development stagnated. By 1200, the crusades[39]

[37]The longest-enduring of which is that between the Sunni and Shia branches of the faith, which began with a disagreement over who should succeed Muhammad himself. Much of the modern-day tension between Saudi Arabia, the Gulf states, and Iran stems from this rupture, which overlaps with the historical ethnic rivalry between Arabs and Persians.

[38]The world's most populous Islamic country is Indonesia.

[39]A series of religious–military campaigns 1096–1291 with the goal of wresting control of the Holy Land, particularly Jerusalem, from Islam.

were well underway and Islam had come to be seen as the enemy of Christian Europe. This feeling only increased over the next few centuries, with Constantinople falling to the Ottomans in 1453. The infusion of knowledge into Europe from Islamic territories around this time helped spur the European renaissance and later scientific revolution. Among European scholars almost to the present day, it has been fashionable to credit Islam merely with the *preservation* of ancient 'European' knowledge; a claim both fanciful and chauvinistic, but plainly stemming from medieval animosity.

Algebra & Algorithms

Proof and axiomatics were learned from Greek texts such as the *Elements.* Like the Greeks, Islamic scholars gave primacy to geometry and proved algebraic relations in a geometric manner.[40] Practical and accurate calculation was more important than it was to the Greeks, and great advances were made in this area. This included completing the development of the Indian decimal place system (hence the dual credit *Hindu–Arabic* numerals).

Beyond our modern numerals, the second most obvious legacy of Islamic mathematics is encountered daily in every mathematics classroom. *Algebra*[41] comes from the Arabic *al-ğabr*, meaning *restoring*. The term originally referred to moving a deficient (negative) quantity from one side of an equation to another. A second term *al-muqābala* (*comparing/balancing*) meant subtracting the same positive quantity from both sides of an equation.

$$\textit{Al-ğabr}: \quad x^2 + 7x = 4 - 2x^2 \implies 3x^2 + 7x = 4$$
$$\textit{Al-muqābala}: \quad x^2 + 7x = 4 + 5x \implies x^2 + 2x = 4$$

Islamic scholars did not use symbols or equations like these; statements were instead written out in sentences.

Muhammad ibn Mūsā al-Khwārizmī (780–850) Born near the Aral Sea in modern Uzbekistan, al-Khwārizmī moved to the center of the empire, eventually rising to become chief librarian at the great school of learning, the *House of Wisdom,* in Baghdad.

Al-Khwārizmī's 820 text, the *Compendious book on the calculation by restoring and balancing*[42] is a synthesis of Babylonian methods and Euclidean axiomatics; an algorithm demonstrated a solution, followed by a geometric proof. After being translated into Latin in the 1100s it became a standard textbook of European mathematics, displacing Euclid in places due to its greater emphasis on practical calculation.

In medieval Europe, the book's name was often shortened simply to *Algebra*, with this term eventually coming to mean all manner of manipulations of equations. The modern term *algorithm* has a similar origin: the Latin *dixit algorismi* literally means *al-Kwārizmī says.*

[40]Like Book II of the *Elements.* Such Greek texts were venerated by Islamic scholars: recognizing the depth of Ptolemy's work on astronomy and trigonometry, they bestowed the name by which it is now known: *Almagest* (the *Great Work*).

[41]Many commonly encountered words beginning *al-* are of Arabic origin (alkali, albatross, etc.). Yet more have been latinized (elixir).

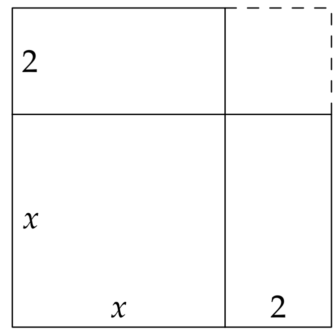
[42]*Al-kitāb al-mukhtasar fī hisāb al-ğabr wa'l-muqābala.*

Example. Here is al-Khwārizmī's approach to the equation $x^2 + 4x = 60$, or, more properly:

> What must be the square which, when increased by four of its roots, amounts to sixty?

The algorithm below may be applied to *any* equation of the form $x^2 + ax = b$ where $a, b > 0$: here a is the number of 'roots,' and b the total 'amount.' A modern, abstract version is in parentheses: note that al-Kwārizmī did not have any of this notation!

- Halve the number of roots: 2 $\left(2 = \frac{1}{2}a\right)$
- Multiply by itself: 4 $\left(4 = \frac{1}{4}a^2\right)$
- Add to the total amount: 64 $\left(64 = \frac{1}{4}a^2 + b\right)$
- Take the root of this: 8 $\left(8 = \sqrt{\frac{1}{4}a^2 + b}\right)$
- Subtract half the number of roots: 6 $\left(6 = \sqrt{\frac{1}{4}a^2 + b} - \frac{a}{2}\right)$



Al-Kwārizmī essentially follows the Babylonian construction of the quadratic formula. His pictorial justification is Euclid's (*Elements*, Thm II. 4). The geometry should be obvious: the original square (x^2) has been increased by four of its roots. The algorithm is, quite simply, 'completing the square' $(x + 2)^2 = 64$.

Other algorithms were supplied to solve every type of quadratic.

A crucial development here is al-Khwārizmī's blending of geometry and numerical algorithm: his approach applies equally to numbers as it does to geometric objects. In some sense this is what modern algebra has become: the abstract manipulation of symbols.

As an example of the power of this idea, consider how Abū Kāmil (Egypt 850–930) generalized Euclid's Book II geometric-algebra arguments to permit substitution, provided the resulting equation was quadratic.

$$\text{If } y = \frac{1+x}{3+x} \text{ and } y^2 + y = 1, \text{ then } x = \sqrt{5}$$

Abū Kāmil essentially substitutes $y = \frac{1+x}{3+x}$ into the quadratic (with solution $y = \frac{\sqrt{5}-1}{2}$). While al-Khwārizmī's methods were geometrically justified, when combined in this fashion the entire process no-longer admits a straightforward geometric interpretation. This method of substitution was an early step towards establishing the modern primacy of algebra and number over geometry and length.

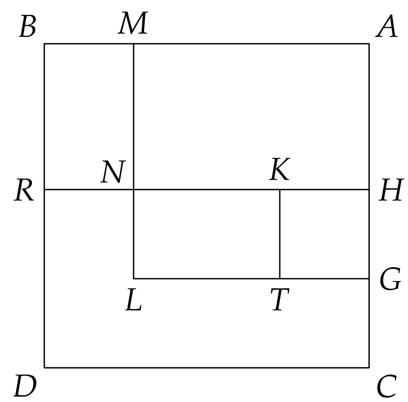
Over the following centuries, this algebraic approach was further improved. In particular, Omar Khayyam (1048–1131) produced ground-breaking work on cubic equations, astronomy, the binomial theorem, and irrational numbers.

Exercises 6.2. *Key concepts: Algebra, Algorithms, Early abstraction*

1. Solve the equations $\frac{1}{2}x^2 + 5x = 28$ and $2x^2 + 10x = 48$ using al-Khwārizmī's methods (first multiply or divide by 2).

2. Al-Khwārizmī gives the following algorithm for solving the equation $bx + c = x^2$.

   - Halve the number of roots.
   - Multiply this by itself.
   - Add this square to the number.
   - Extract the square root.
   - Add this to half the roots.



   Translate this into a formula. Then give a geometric argument for the validity of the approach using the picture: $\overline{HC}$ has length b where G is the midpoint, rectangle $\square ABRH$ has area c, $\square KHGT$ and $\square AMLG$ are squares, and the large square $\square ABDC$ has side-length x.

3. Solve the following problems by Abū Kāmil (use modern algebra!).

   (a) Suppose 10 is divided into two parts and the product of one part by itself equals the product of the other part by the square root of 10. Find the parts.

   (b) Suppose 10 is divided into two parts, each of which is divided by the other, and the sum of the quotients equals the square-root of 5. Find the parts.

6.3 Islamic Mathematics II: Spherical Trigonometry and the *Qibla*

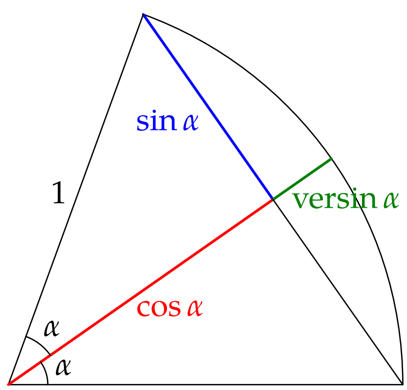
Late 8$^{\text{th}}$ century Indian work on trigonometry, linking back to Hipparchus, was known in Baghdad, as was the work of Ptolemy. While Islamic scholars were certainly interested in astronomy, their study of trigonometry had a more practical, if still religious, purposes. While at prayer, Muslims face the Ka'aba in the Great Mosque at Mecca: this is the *qibla* (*direction* in Arabic). Mosques are typically built so that one wall faces Mecca for convenience; if this is not possible, an arrow indicating the *qibla* might be placed in an alcove. In Muhammad's time (when Muslims faced Jerusalem not Mecca), determining the *qibla* was relatively easy, though as Islam spread the curvature of the earth made determination more difficult. The religious impetus behind this problem motivated the development of spherical trigonometry: the methods developed are still used today for the purposes of navigation.

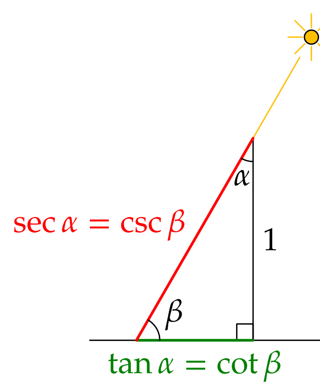
Terminology and Trigonometric Tables

Scholars worked with the Indian *half-chord* (sine), and with circles of various radii. Al-Battānī (c. 858–929) introduced an early version of *cosine* as the *complementary half-chord* for acute angles, and an analogue of the modern function *versine*:[43]

$$\operatorname{versin}\theta = 1 - \cos\theta$$

Al-Bīrūnī (973–1048) defined versions of tangent, cotangent, secant and cosecant by projecting (e.g., a sundial) onto either a horizontal or a vertical plane. In the second picture below, the gnomon is the vertical stick of length 1. With this definition, al-Bīrūnī moves towards the modern consideration of trigonometry in terms of *triangles* rather than circles.





Trigonometric tables with improved accuracy over Ptolemy were created for all these 'functions.' Abū al-Wafā (940–998) and his descendants computed sine & tangent values for every minute of arc accurate to *five sexagesimal places* (one part in 777 million!) via repeated applications of the half-angle formula and interpolating using the downwards concavity of the sine function (draw a picture!):

$$\sin(\alpha+\beta) - \sin\alpha < \sin\alpha - \sin(\alpha-\beta) \qquad \text{whenever} \qquad 0^\circ < \alpha-\beta < \alpha+\beta < 90^\circ$$

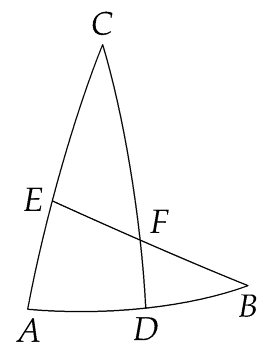
[43] *Versed sine* refers to the measurement of a length in a re*versed* direction (perpendicular) to that of sine.

Calculating the *Qibla* I: Al-Wafā and the Sine Rule

In what follows we observe several conventions:

- A single letter A refers to a *point* or to the *angle measure* in a triangle with vertex A.
- $\overset{\frown}{AB}$ means the *great-circle arc* joining points A, B or its *arc-length*. A *spherical triangle* $\triangle ABC$ comprises three points on a sphere joined by great-circle arcs.
- $\overline{AB}$ means the *straight line-segment* joining points A, B with *length* $|AB|$.
- We use modern language and work on the unit sphere (center O). The arc-length along a great-circle thus equals the central angle subtended by that arc in radians: $\overset{\frown}{AB} = \measuredangle AOB$.

Ptolemy and the Indians had already done some relevant work, though Ptolemy's approach relies heavily on Menelaus' Theorem (c. AD 100).



Theorem (Menelaus). *For the pictured configuration of spherical triangles on a sphere of radius 1,*

$$\frac{\sin \overset{\frown}{CE}}{\sin \overset{\frown}{AE}} = \frac{\sin \overset{\frown}{CF}}{\sin \overset{\frown}{DF}} \cdot \frac{\sin \overset{\frown}{BD}}{\sin \overset{\frown}{AB}}$$

Applying Menelaus is difficult since one typically needs to create many new spherical triangles. Al-Wafā simplified matters with an alternative result.

Theorem (Al-Wafā). *If $\triangle ABC$ and $\triangle ADE$ are spherical triangles with right angles at B, D and a common acute angle at A, then*

$$\frac{\sin \overset{\frown}{BC}}{\sin \overset{\frown}{AC}} = \frac{\sin \overset{\frown}{DE}}{\sin \overset{\frown}{AE}}$$

In fact these ratios equal $\sin \alpha$ where α is the angle at A, though al-Wafā didn't say this.

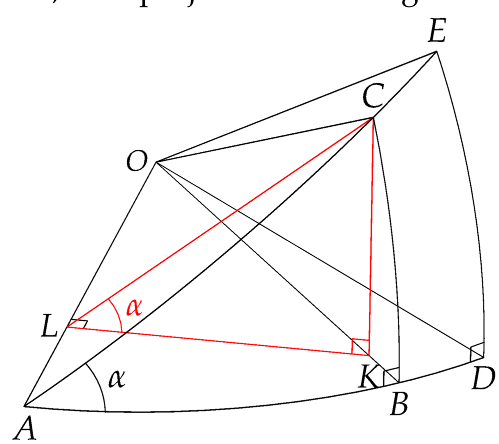
Proof. Project C onto the plane containing O, A, B to produce K, then project K to $\overline{OA}$ to get L. Consider the planar right-triangle $\triangle CKL$. Since α is the angle between two planes, we have $\alpha = \measuredangle CLK$. Moreover

$$|CK| = \sin \measuredangle COK = \sin \measuredangle COB = \sin \overset{\frown}{BC}$$
$$|CL| = \sin \measuredangle COL = \sin \measuredangle COA = \sin \overset{\frown}{AC}$$

The usual sine formula for plane triangles says

$$\sin \alpha = \frac{|CK|}{|CL|} = \frac{\sin \overset{\frown}{BC}}{\sin \overset{\frown}{AC}}$$

The same ratio is obtained for $\triangle ADE$.



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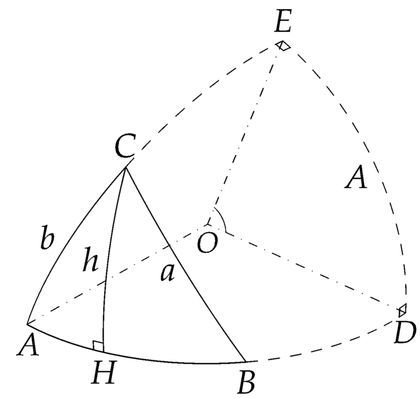
The spherical sine rule quickly follows. Here is al-Wafā's argument (the modern approach is a little simpler, see Exercise 6). For the pictured triangle $\triangle ABC$:

- Extend $\widehat{AB}$ and $\widehat{AC}$ to quarter-circles $\widehat{AD} = 90° = \widehat{AE}$ so that $\triangle ADE$ has right-angles at D and E.
- Since $ACEO$ and $ABDO$ define planes through the center of the sphere meeting at angle A, it follows that $\widehat{DE}$ is an arc subtending the central angle A: that is $\widehat{DE} = A$.
- Drop the perpendicular from C to $H \in \widehat{AD}$ and apply al-Wafā's theorem to the right-triangles $\triangle AHC$ and $\triangle ADE$:



$$\frac{\sin h}{\sin b} = \frac{\sin \widehat{DE}}{\sin \widehat{AE}} = \frac{\sin A}{1} \implies \sin h = \sin b \sin A$$

Mirroring this construction on the $\overline{AC}$ side of $\triangle ABC$ yields $\sin h = \sin a \sin B$. Equating the $\sin h$ terms produces the first part of the main result.

Corollary (Sine rule). *If a, b, c are the side-lengths of a spherical triangle $\triangle ABC$, then*

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

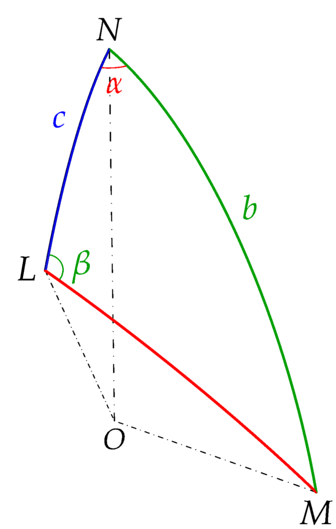
The final equality comes from dropping a second altitude, say from B. Armed with the sine rule, al-Wafā could solve spherical triangles and thus compute the *qibla.* We omit his complete method, since it required several auxiliary triangles and is difficult to follow.

Calculating the *Qibla* II: Al-Bīrūnī and the Cosine Rule

Al-Bīrūnī further simplified matters by developing what is essentially the cosine rule. We apply his method to find the *qibla* from a location L (remember that our sphere has radius 1).

Let M be Mecca and N the north pole. The *qibla* is β, the initial bearing from L to M. Our known data are the latitudes and longitudes of L and M, specifically:

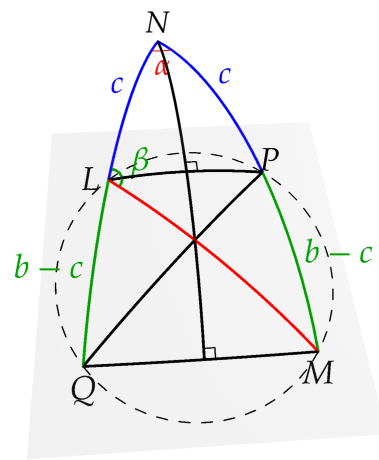
- The angle α is the difference in the longitudes of L and M.
- The arc-lengths b, c are the *colatitudes*[44] of M and L respectively.



We have SAS data for $\triangle NLM$. If this were a planar triangle, we'd use the cosine rule to compute $\widehat{LM}$ and then the sine rule to find β. Al-Bīrūnī does essentially this on the sphere...

[44]The *colatitude* of a point, 90° minus its latitude, is measured southwards from the north pole. Since our model sphere has radius 1, the arc-lengths b, c equal the colatitudes in radians.

The spherical cosine rule follows from Ptolemy's Theorem (pg. 56) and, for us, a translation to modern notation and some trigonometric identities.



- Let $P \in \widehat{NM}$ have the same latitude as L and extend $\widehat{NL}$ to Q with the same latitude as M.
- By symmetry, L, P, Q, M are *coplanar*. The quadrilateral $LPQM$ lies on the intersection of a plane and a sphere: a *circle*! Ptolemy's Theorem therefore applies:

  $$|LM|\,|PQ| = |LQ|\,|PM| + |LP|\,|QM|$$

  where distances are measured as straight lines (chords).
- By symmetry ($|PQ| = |LM|$ and $|LQ| = |PM|$), this becomes

  $$|LM|^2 = |LQ|^2 + |LP|\,|QM|$$

- The great-circle arc-lengths on the sphere may be found via the usual chord relations: e.g.,

  $$|LM| = \operatorname{crd} \widehat{LM} = 2\sin\frac{\widehat{LM}}{2}$$

  whence Ptolemy's theorem becomes a relation between *arc-lengths*

  $$\sin^2\frac{\widehat{LM}}{2} = \sin^2\frac{b-c}{2} + \sin\frac{\widehat{LP}}{2}\sin\frac{\widehat{QM}}{2}$$

- Bisect α to obtain two pairs of right-triangles and apply al-Wafā's theorem:

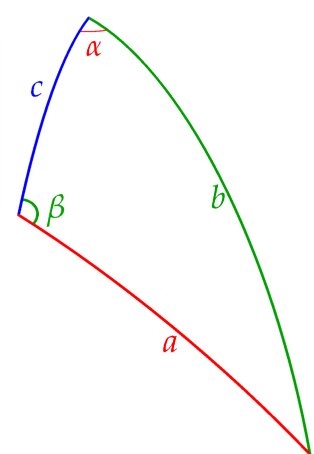
  $$\sin\frac{\alpha}{2} = \frac{\sin\frac{\widehat{LP}}{2}}{\sin c} = \frac{\sin\frac{\widehat{QM}}{2}}{\sin b} \implies \sin^2\frac{\widehat{LM}}{2} = \sin^2\frac{b-c}{2} + \sin^2\frac{\alpha}{2}\sin c\sin b \qquad (*)$$

- To complete the proof we apply the multiple-angle formulæ

  $$\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x) \qquad \cos(b-c) = \cos b\cos c + \sin b\sin c$$

Corollary (Cosine rule). *In a spherical triangle with sides a, b, c and angle α opposite a, we have*

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha$$



For our triangle of interest, $a = \widehat{LM}$. Given points L, M (and thus b, c, α), one uses the cosine rule to compute a and then the sine rule to find the *qibla* β (whew!):

$$\frac{\sin b}{\sin\beta} = \frac{\sin a}{\sin\alpha} \implies \sin\beta = \frac{\sin\alpha\sin b}{\sin a}$$

Example. For fun, here is some real-world data. The longitude and latitude of London and Mecca are, respectively 51°30′ N, 8′ W, and 21°25′ N, 39°49′ E. This corresponds to

$$\alpha = 39°49' + 8' = 39°57'$$
$$b = 90° - 21°25' = 68°35'$$
$$c = 90° - 51°30' = 38°30'$$

By al-Bīrūnī's cosine rule,

$$\cos a = \cos 68°35' \cos 38°30' + \sin 68°35' \sin 38°30' \cos 39°57' \implies a = 43.110°$$

Since Earth's circumference is 24900 miles, the distance London → Mecca is $\frac{43.110 \times 24900}{360} = 2981$ miles. Al-Wafā's sine rule computes the *qibla*

$$\beta = 180° - \sin^{-1} \frac{\sin \alpha \sin b}{\sin a} = 118°59'$$

where we subtracted from 180° since the relevant angle is plainly obtuse. Check it yourself at the Great Circle Mapper (the website uses airports for slightly different initial data).

Spherical Trigonometry Cheat Sheet

Let $\triangle ABC$ be a spherical triangle with side-lengths a, b, c on a sphere of radius 1.

Basic trigonometry. If $\triangle ABC$ is right-angled at C

$$\sin A = \frac{\sin a}{\sin c} \qquad \cos A = \frac{\tan b}{\tan c} \qquad \tan A = \frac{\tan a}{\sin b}$$

Al-Wafā essentially proved the first; the others follow from trig identities ($\cos^2 A = 1 - \sin^2 A \ldots$)

Sine rule (Al-Wafā)

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Cosine rule (Al-Bīrūnī)

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

The spherical Pythagorean Theorem is the special case $\cos c = \cos a \cos b \;\; (C = 90°)$.

If the sphere has radius r, divide all arc-lengths by r before applying the results, e.g.,

$$\sin A = \frac{\sin(a/r)}{\sin(c/r)}$$

As $r \to \infty$, we have $\sin \frac{a}{r} \approx \frac{a}{r}$ and $\cos \frac{a}{r} \approx 1 - \frac{a^2}{2r^2}$, which recover the flat (Euclidean geometry) versions of these statements.

Examples. Here are two simple examples of spherical trigonometry at work.

1. On a sphere of radius 1, an equilateral triangle has side length $\frac{\pi}{3}$. Splitting it in half creates two right-triangles with adjacent $\frac{\pi}{6}$ and hypotenuse $\frac{\pi}{3}$. The angles in the original equilateral triangle are therefore

$$\alpha = \cos^{-1}\frac{\tan\frac{\pi}{6}}{\tan\frac{\pi}{3}} = \cos^{-1}\frac{1}{3} \approx 70.53^\circ$$

The angle sum in the triangle is $\Sigma_\triangle = 3\alpha \approx 211.59^\circ$!

2. An airfield is at C and two planes are at A and B. The bearings and distances to the aircraft are 45°, 2000 miles, and 90°, 4000 miles respectively. Find the distance between the aircraft.

This is just the cosine rule! We have a spherical triangle with sides 2000 and 4000 separated by 45°. If $r \approx 4000$ miles is Earth's radius, then

$$\begin{aligned}\cos\frac{c}{r} &= \cos\frac{2000}{r}\cos\frac{4000}{r} + \sin\frac{2000}{r}\sin\frac{4000}{r}\cos 45^\circ \\ &= \cos\frac{1}{2}\cos 1 + \frac{1}{\sqrt{2}}\sin\frac{1}{2}\sin 1 \\ \implies c &= 2833 \text{ miles}\end{aligned}$$

This is a little closer than the value (2947 miles) we'd obtain from assuming a flat Earth.

Modern calculations use a slightly different, though equivalent, approach to minimize the error inherent in estimating cosine for small values: look up the *haversine formula* if you're interested.

Exercises 6.3. *Key concepts: Improved trigonometric tables and functions, Spherical trigonometry* (qibla *and navigation*)

1. A right-isosceles triangle on the surface of a unit sphere has equal legs of length $\frac{\pi}{4}$. Find the length of the hypotenuse and the sum of the angles in the triangle.

2. Explain the observation on page 77, that the downwards concavity of the sine function proves

$$0^\circ < \alpha - \beta < \alpha + \beta < 90^\circ \implies \sin(\alpha+\beta) - \sin\alpha < \sin\alpha - \sin(\alpha - \beta)$$

3. Suppose we have a spherical triangle (sphere radius 1) with data

$$c = 30^\circ, \quad b = 60^\circ, \quad \alpha = 60^\circ$$

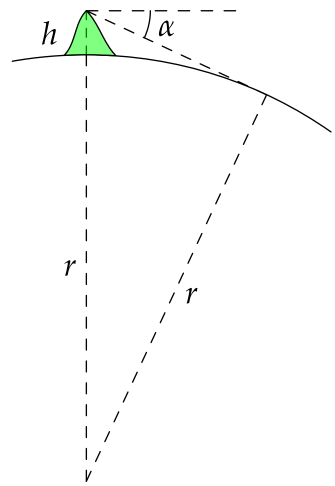
   (a) Use the cosine rule to find a.

   (b) Compute the remaining angles in the triangle. What do you observe about the sum of the angles $\alpha + \beta + \gamma$?

4. Determine the *qibla* for Rome (latitude 41°53′ N, longitude 12°30′ E).

   Repeat for the UCI campus (33°39′ N, 117°51′ W).

5. Al-Bīrūnī devised a method for determining the radius r of the earth by sighting the horizon from the top of a mountain of known height h. He would measure α, the angle of depression from the horizontal to which one sights the apparent horizon. Show that

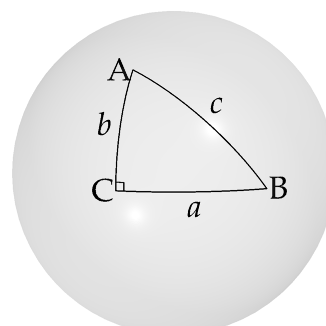


$$r = \frac{h\cos\alpha}{1-\cos\alpha}$$

   In a particular case, al-Bīrūnī measures $\alpha = 34'$ from a mountain of height $652;3,18$ cubits. Assuming that a cubit equals $18''$, convert your answer to miles and compare with the modern value. Discuss the efficacy of this method.

6. Recall in al-Wafā's Theorem that the ratio equals the sine of the angle at A. Use this observation on $\triangle ABC$ with the dropped perpendicular $\overline{CH}$ to obtain a cheaper proof of the sine rule.

7. On a sphere of radius r, Pythagoras' Theorem may be stated



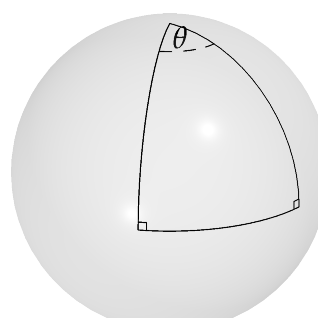
$$\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r}$$

   where c is the hypotenuse and a, b the other side-lengths.

   (a) Use the Maclaurin series $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ to expand this result to degree 4.

   (b) Suppose a, b are constant so that c is a function of r. Prove that $\lim_{r\to\infty} c^2 = a^2 + b^2$. Why does this make sense?

8. Construct a triangle on the surface of a sphere of radius r by taking two lines of longitude making an angle θ from the north pole to the equator. Prove that the area of the triangle is



$$A = r^2\theta$$

   What does Pythagoras' (Exercise 7) say for this triangle?

   (*Hint: What fraction of the sphere is covered by the triangle?*)