7 The Renaissance in Europe

7.1 Transmission of Knowledge and Notational Development

Between the fall of Rome in 476 and the early renaissance[45] (c. 1100) came Europe's dark ages. The simplistic view is that this was a time of limited learning and technological progress, though reality was more complex. Learning and scholarship persisted in monasteries, though new research took second place to a conservative focus on preserving the wisdom of the ancients.

By 1100, some smaller kingdoms were starting to come together in more stable arrangements. The maps below show the consolidation of the major feudal societies[46] of medieval Europe: France, England, the Holy Roman Empire, Poland, Austria, etc. While it would have meant little to most peasants, a large stable nation can support a significant elite population with the time and money to pursue education, and fund libraries and universities. By the 1700s, the borders of western Europe are largely recognizable; political and social organization had expanded so that a much larger proportion of the population—though small in comparison to today—could take advantage of and contribute to the growth of knowledge. The European renaissance is often contrasted with a decline in Islamic power, but again the story is more complex: Islam retreats from Spain while the Ottoman Empire becomes dominant in south-east Europe.

Between 900 and 1300, the population of Europe roughly tripled to around 100 million. Trade increased, along with which came knowledge.[47] Learning (and land) came via wars with Islam (the Crusades, Spain, etc.). It helped that Islamic scholars so venerated the Greeks; Europeans could tell themselves that they were merely 'reclaiming' ancient knowledge which had been 'stolen' by their cultural and religious enemies. The fall of Constantinople to Mehmet in 1453 marks both the high point of European Islamic conquest and the start of the decline of Islamic scientific dominance. Many intellectuals fled Constantinople—where the Byzantines preserved much of Alexandria's learning and were more exposed to eastern ideas—for Rome, helping to further fuel developments. With powerful enemies to the east, Europeans began travelling greater distances by sea,[48] beginning the colonial era of global European empire.

[45]Literally *rebirth.* Dates vary by location and discipline (Italy vs. France, art vs. philosophy, etc.) but a wide net would encompass the 12$^{\text{th}}$ to 17$^{\text{th}}$ centuries.

[46]An arrangement where powerful landowners could demand service (rent, food, warriors) from their tenants.

[47]Venice was a particularly important trading hub. From here, Marco Polo (1254–1324), perhaps the most famous trader of the period, travelled the silk road to China.

[48]Christopher Columbus (born Genoa 1451) 'discovered' America in 1492 while looking for sea routes to Asia.

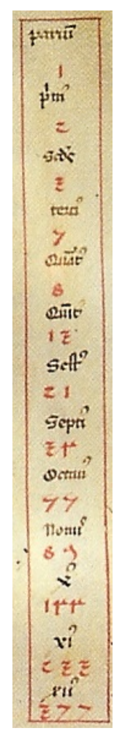
Education in the Renaissance

Scientific and philosophical progress was spurred by the translation of works from Arabic and Greek into Latin, with the first universities being formed to teach this canon: Bologna 1088, Paris 1150, and Oxford 1167. The typical student was a young man of wealth who had been privately tutored in grammar, logic & rhetoric (the *trivium*). At university he would study the Greek-influenced *quadrivium* (geometry, astronomy, arithmetic & music). While Islamic improvements were incorporated, scholars gave pre-eminent credit to the Greeks: Euclid for geometry, Aristotle for logic/physics, Hippocrates/Galen for medicine, Ptolemy for astronomy. Early universities were often funded by the Church and 'research,' was more likely to involve the justification of biblical passages using Aristotle than the conduct of experiments.

Leonardo da Pisa (Fibonacci c. 1175–1250)

Fibonacci[49] likely first encountered the Hindu–Arabic numerals while trading with his father in North Africa. He was impressed by the ease of calculation they afforded and is the first European known to use them (contemporary Europeans used Roman numerals and Egyptian fractions). Fibonacci's 1202 text *Liber Abaci* was written to instruct traders in their use. The first page below explains how to compute with decimal fractions, with the two columns at the bottom of the page showing how to repeatedly multiply 100 (and then 10) by the fraction $\frac{9}{10}$. Thus:

$$100, \quad 90, \quad 81, \quad \frac{9}{10}72\ (=72.9), \quad \frac{1}{10}\frac{6}{10}65\ (=65.61), \quad \frac{9}{10}\frac{4}{10}\frac{0}{10}59\ (=59.049), \quad \text{etc.}$$



[49]The name was given to Leonardo by French scholars of the 1800s: *filius Bonacci* means *son of Bonacci.*

Note how the fractional part is written in reverse (relative to modern practice) on the left, using a bar to separate numerator and denominator: the Indians wrote fractions without a bar and it is thought Islamic scholars inserted it for clarity in the 1100s.

The second picture is of Fibonacci's famous sequence, which he illustrates using the famous problem of breeding rabbits. Read top-to-bottom: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. While named for him, the sequence had been discussed for at least 1000 in India. Amongst other inheritances from the Hindu–Arabic tradition, Fibonacci is the first known European to work with negative numbers, provided these represented deficiencies or debts in accounting.

Algebraic Notation and Development

To a modern reader, the most obvious mathematical development of the renaissance is notational. Hindu–Arabic numerals and fractional notation were cutting-edge for the 1200s, but essentially everything else in Fibonacci's work was written in sentences. Over the next 500 years, notational improvements developed which eventually allowed algebra to eclipse geometry as the primary language of reasoning. Here is a very brief summary.

Italian Abacists, 14$^{\text{th}}$ C. This group continued Fibonacci's advocacy for the Hindu–Arabic system against the traditional use of Roman numerals, and also for the use of accompanying algorithms. Their approach was highly practical and applied largely for the conduct of trade. Here is a typical problem described by the group:

> The *lira* earns three *denarii* a month in interest. How much will sixty *lire* earn in eight months?

The Abacists introduced various shorthands for unknowns and some mathematical operations: *cosa* ('thing') was used for an unknown; *censo, cubo* and *radice* meant, respectively, square, cube and (square-)root. These expressions could be combined, for example 'ce cu' (read *censo di cubo*) referred to the sixth power of an unknown $(x^3)^2 = x^6$.

Luca Pacioli, Italy late 1400s. Introduced $\overline{p}, \overline{m}$ (*piu, meno*) for plus and minus. For example $8\overline{m}2$ denoted eight minus two.

Nicolas Chuquet, France 1484. His text *Triparty en la science des nombres* borrowed Pacioli's $\overline{p}$ and $\overline{m}$, and introduced an R-notation for roots. For instance R^47 meant $\sqrt[4]{7}$, while

$$\sqrt[5]{4-\sqrt{2}} \quad \text{would be written} \quad R^5\underline{4\overline{m}R2}$$

Chuquet underline terns to indicate grouping in the way we now use parentheses.

Christoff Rudolff, Vienna 1520s. Introduced symbols similar to x and ζ for an unknown and its square. He had other symbols for odd powers and produced tables showing how to multiply these. The words he used for these symbols show Italian and French influence: algebra in the German-speaking world was known as the art of the *coss* (German for *thing*).

By Rudolff's time, the $\pm$ symbols had been in use for around 30 years as a prefix denoting an excess or deficiency in a quantity (profit/loss in accounting); Rudolff began to use them as *algebraic operations* instead of $\overline{p}$ and $\overline{m}$. A period denoted equals, and he is also credited with the first use of the square-root sign $\surd$, which is nothing more than a stylized lower-case r. This would have been written *in front* of a number, e.g. $\surd 23$ rather than $\sqrt{23}$.

Robert Recorde, England 1557. Introduced the equals sign in *The Whetstone of Witt,* asserting that, 'No two things are more equal than a pair of parallel lines.'

Francois Viète, France 1540–1603. Before Viète, mathematicians typically described how to solve particular equations algorithmically via examples: e.g. $x^3 + 3x = 14$ rather than the general form $x^3 + bx = c$. Readers were expected to change numbers to fit a required situation while following the same algorithmic structure. Viète pioneered the modern use of abstract constants, using letters to represent both unknowns and constants. As we'll discuss shortly, this allowed him to investigate how the roots of an equation depend on its constants, a key step in abstract thinking.

Simon Stevin, Holland 1548–1620. *De Thiende* (The Tenths) demonstrated how to calculate using decimals rather than fractions. Stevin arguably completed the journey whereby the concept of *number* subsumed that of *magnitude.* This increased the application of algebra by permitting the numerical description of any geometric magnitude.

William Oughtred, England 1575–1660. Introduced × for multiplication, though he often simply used juxtaposition. Oughtred combined Viète's general approach (abstract constants) with symbolic algebra. For instance, he'd write a quadratic equation as $A_q + BA + C = 0$, where A_q means 'A-squared' (A-*quadratum* in Latin) and B, C are constants. Its solution, the quadratic formula, would be written

$$A = \surd : \frac{1}{4}B_q - C : -\frac{1}{2}B$$

In Oughtred's notation, colons were parentheses.

Thomas Harriot, England 1560–1621. Made several steps towards modern notation including juxtaposition for multiplication and a modern *encompassing* root-sign. For example,

$$\sqrt[4]{cccc + 27aa\sqrt[3]{2+b}} \quad \text{meant} \quad \sqrt[4]{c^4 + 27a^2\sqrt[3]{2+b}}$$

René Descartes, France/Holland 1596–1650. Used exponents for powers (a^2, a^3) and solidified the convention of using letters at the end of the alphabet (x, y, z) for unknowns and those at the beginning (a, b, c) for constants.

While modern mathematics uses many specialized symbols ($\emptyset$, $\Rightarrow$, e, π, etc.), basic notation is essentially unchanged from the mid 1600s. This is not to say that all mathematicians uniformly used the most modern notation available: for instance, papers of Leonhard Euler (1700s) used Harriot's juxtaposition notation for exponents, and some published works from the late 1800s still wrote equations in words.

In this context, it is also worth mentioning Gutenberg's 1439 invention of the printing press. The relative ease of production meant a great increase in the availability of written material. This naturally aided the dissemination of all learning, but led also to the abandonment of some older texts when money could not be found to make an updated printed edition.

Exercises 7.1. *Key concepts: Introduction of Hindu–Arabic numerals to Europe, Notational Development 1200–1650*

1. Repeatedly divide 10 by 5 a total of five times (to $\frac{10}{5^5}$), expressing the results using Fibonacci's notation.

2. What would Nicolas Chuquet have meant by the expression $4\bar{p}R^3\underline{7\ \bar{m}R5}$?

3. How would William Oughtred have expressed the solution to the quadratic equation $A_q = BA + C$? What about Thomas Harriot?

4. I am owed 3240 *florins.* The debtor pays me 1 *florin* the first day, 2 the second day, 3 the third say, etc. How many days does it take to pay off the debt?

5. (A problem of Antonio de Mazzinghi) Find two numbers such that multiplying one by the other makes 8 and the sum of their squares is 27.

   (*Hint: let the numbers be* $x \pm \sqrt{y}$)

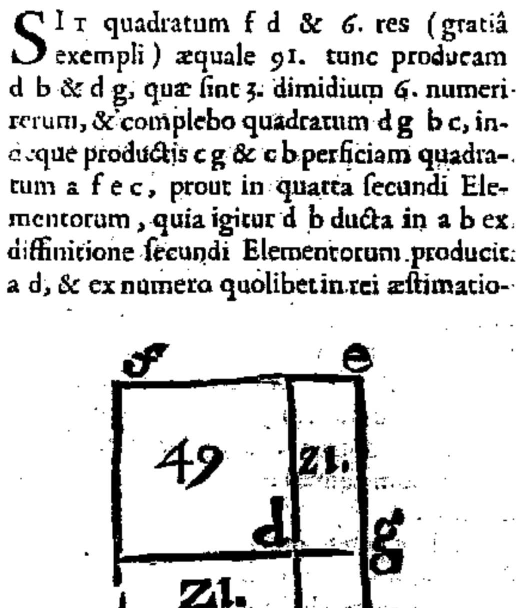
7.2 Polynomials: Cardano, Factorization & the Fundamental Theorem

As an example of contemporary algebraic notation and practice, we consider Girolamo Cardano's 1545 *Ars Magna* (*Great Art or the Rules of Algebra*), in which he describes how to solve quadratic and cubic equations.

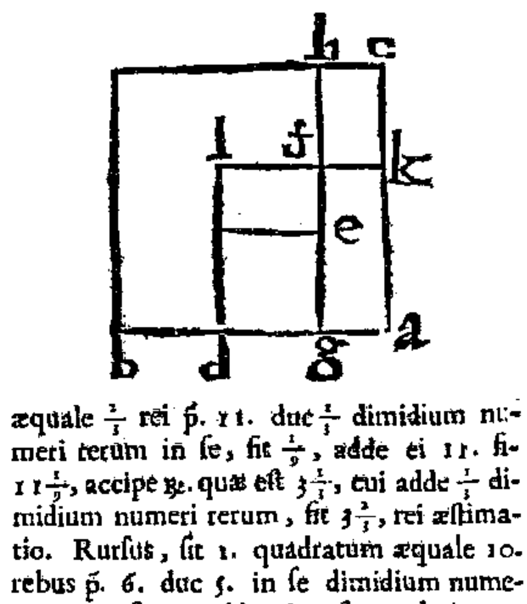
The first example below (from *Caput V*) is the beginning of Cardano's description of how to solve the equation $x^2 + 6x = 91$ by completing the square: the Latin *quadratum & 6 res aequalie 91* translates as "square and 6 things equals 91." He employs several single-letter abbreviations but still writes in sentences and provides a pictorial justification. In modern algebra, his argument is nothing more than completing the square:

$$\begin{aligned} x^2 + 6x = 91 &\implies x^2 + 2 \cdot 3x + 3^2 = 91 + 3^2 \\ &\implies (x+3)^2 = 100 \\ &\implies x + 3 = 10 \\ &\implies x = 7 \end{aligned}$$

where the picture justifies $7^2 + 2 \cdot 21 + 3^2 = 10^2$.


SIT quadratum f d & 6. res (gratiâ exempli) æquale 91. tunc producam d b & d g, quæ ſint 3. dimidium 6. numeri rerum, & complebo quadratum d g b c, indeque productis c g & c b perficiam quadratum a f e c, prout in quarta ſecundi Elementorum, quia igitur d b ducta in a b ex diffinitione ſecundi Elementorum producit a d, & ex numero quolibet in rei æſtimatio-

Completing the square



æquale $\frac{1}{3}$ rei p̄. 11. duc $\frac{1}{3}$ dimidium numeri rerum in ſe, fit $\frac{1}{9}$, adde ei 11. fit 11 $\frac{1}{9}$, accipe R̸. quæ eſt 3 $\frac{1}{3}$, cui adde $\frac{1}{3}$ dimidium numeri rerum, fit 3 $\frac{2}{3}$, rei æſtimatio. Rurſus, ſit 1. quadratum æquale 10. rebus p̄. 6. duc 5. in ſe dimidium numeri rerum, fit 25. adde ei 6. fit 31. huius R̸. adde 5. dimidium numeri rerum, erit rei æſtimatio, R̸. 31. p̄. 5. Rurſus fit 1. qua-

More quadratic equations

The quadratic algorithms were well-established by this time. On the following page is a picture (second above) reminiscent of al-Khwārizmī (Exercise 6.2.2), after which comes further mathematical notation justifying the solution of another quadratic equation. Even though Rudolff's $\pm$ and $\surd$ were in use, Cardano wrote almost everything in words augmented with fractional notation and Pacioli's $\overline{p}$ and $\overline{m}$.

As was typical for the time, Cardano describes negative solutions as fictitious; he even writes the square root of -15 at one point, though only to mention its absurdity. He also follows the

Islamic approach of solving a concrete problem of each type rather than proceeding abstractly, though observe the *gratia exempli* ("for the sake of an example") as acknowledgement that the general problem $x^2 + ax = b$ $(a, b > 0)$ may be solved identically.

The Cubic Formula

It is for the solution of cubic (and quartic) equations that Cardano is most famous. Below we describe, in modern notation, Cardano's method for solving the cubic equation $x^3 + bx = c$ where $b, c > 0$, though we stress (again) that he gave only *examples* and not a general formula.

Let u, v satisfy $u^3 - v^3 = c$ and $uv = \frac{b}{3}$. Then $x = u - v$ is seen to solve the cubic:

$$\begin{aligned} x^3 + bx &= (u-v)^3 + b(u-v) \\ &= u^3 - 3u^2v + 3uv^2 - v^3 + b(u-v) \\ &= (u^3 - v^3) + (u-v)(b - 3uv) = c \end{aligned}$$

However u and v also satisfy

$$(u^3 + v^3)^2 = (u^3 - v^3)^2 + 4(uv)^3 = c^2 + 4\left(\frac{b}{3}\right)^3$$

We therefore obtain a system of linear equations in the unknowns u^3, v^3. These are easily solved:

$$\begin{cases} u^3 + v^3 = \sqrt{c^2 + 4\left(\frac{b}{3}\right)^3} \\ u^3 - v^3 = c \end{cases} \implies \begin{cases} u = \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} + \frac{c}{2}} \\ v = \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} - \frac{c}{2}} \end{cases}$$

$$\implies x = u - v = \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} + \frac{c}{2}} - \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} - \frac{c}{2}}$$

Examples Here are three applications of Cardano's method.

1. The equation $x^3 + 6x = 7$ has the obvious solution $x = 1$. Indeed,

$$\begin{cases} u^3 - v^3 = 7 \\ uv = \frac{6}{3} = 2 \end{cases} \implies (u^3 + v^3)^2 = 7^2 + 4 \cdot 2^3 = 81 \implies u^3 + v^3 = 9$$

$$\implies u^3 = \frac{1}{2}(9+7) = 8, \qquad v^3 = \frac{1}{2}(9-7) = 1$$

$$\implies x = u - v = \sqrt[3]{8} - \sqrt[3]{1} = 2 - 1 = 1$$

2. This time we solve $x^3 + 3x = 14$.

$$\begin{cases} u^3 - v^3 = 14 \\ uv = \frac{3}{3} = 1 \end{cases} \implies (u^3+v^3)^2 = 14^2 + 4\cdot 1^3 = 200 \implies u^3 + v^3 = 10\sqrt{2}$$
$$\implies \begin{cases} u^3 = \frac{1}{2}(10\sqrt{2}+14) = 5\sqrt{2}+7 = (\sqrt{2}+1)^3 \\ v^3 = 5\sqrt{2}-7 = (\sqrt{2}-1)^3 \end{cases}$$
$$\implies x = u - v = (\sqrt{2}+1) - (\sqrt{2}-1) = 2$$

As this example shows, Cardano's formula might produce an ugly expression for a simple answer—of course, the cube root of $5\sqrt{2} \pm 7$ is the tip of everyone's tongue!

3. Finally, we consider $x^3 + 3x = 10$. The method works, though in this case the answer is just ugly.

$$\begin{cases} u^3 - v^3 = 10 \\ uv = \frac{3}{3} = 1 \end{cases} \implies (u^3+v^3)^2 = 10^2 + 4\cdot 1^3 = 104 \implies u^3 + v^3 = 2\sqrt{26}$$
$$\implies u^3 = \sqrt{26}+5, \qquad v^3 = \sqrt{26}-5$$
$$\implies x = u - v = \sqrt[3]{\sqrt{26}+5} - \sqrt[3]{\sqrt{26}-5} \approx 1.6989$$

Being unable or unwilling to work directly with negative numbers, Cardano modified his method to solve other cubics such as $x^3 + c = bx$, and moreover described how to remove a quadratic term from a cubic using what is now known as the *Tschirnhaus substitution* ($x = y - \frac{a}{3}$):

$$x^3 + ax^2 + bx + c = \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = y^3 - \frac{a^2}{3}y + \cdots \qquad (*)$$

Cardano's student, Lodovico Ferrari, extended the method to solve quartic equations in terms of the solution of a resultant cubic.

Negative Solutions and Complex Numbers

By the late 1500s, mathematicians were mentioning negative solutions to equations. These solutions were usually described as *fictitious,* or *false roots,* but this didn't always prevent them from being investigated. Rafael Bombelli (1526–1572, Rome) even introduced a notation for complex numbers, described their algebra, and showed how they could be used to find solutions to any quadratic or cubic equation. For instance, he wrote $4 + 3i$ and $4 - 3i$ as follows:

> 4 *p di m* 3, read '*quattro piu di meno tre*' (four plus of minus three), and,
> 4 *m di m* 3, '*quattro meno di meno tre.*'

Given Bombelli's belief in the fictitiousness of complex numbers, the effort he expended in their honor is extraordinary: this is a prime example of pure abstraction, math for the sake of math!

In modern language, the three roots of Cardano's cubic $x^3 + bx = c$ are

$$u - v, \quad \zeta u - \zeta^2 v, \quad \zeta^2 u - \zeta v$$

where $\zeta = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}i}{2}$ is a primitive cube root of unity. Together with the Tschirnhaus substitution $(*)$, Cardano's formula therefore solves all cubic equations.

Examples We first return to one of our previous examples, then consider a substitution.

1. Recall that if $x^3 + 3x = 14$, then Cardano's method returns $u = \sqrt{2} + 1$ and $v = \sqrt{2} - 1$. The three solutions to the cubic are therefore

$$\begin{aligned} &u - v = 2 \\ &\zeta u - \zeta^2 v = (\sqrt{2}+1)\frac{-1+\sqrt{3}i}{2} - (\sqrt{2}-1)\frac{-1-\sqrt{3}i}{2} = -1 + \sqrt{6}i \\ &\zeta^2 u - \zeta v = (\sqrt{2}+1)\frac{-1-\sqrt{3}i}{2} - (\sqrt{2}-1)\frac{-1+\sqrt{3}i}{2} = -1 - \sqrt{6}i \end{aligned}$$

   As a sanity check, it is worth checking these solutions directly.

2. To find a solution to $x^3 + 3x^2 = 3$, we first perform the Tschirnhaus substitution Define a new variable y via $x = y - \frac{3}{3} = y - 1$, then

$$y^3 - 3y^2 + 3y - 1 + 3(y^2 - 2y + 1) = 3 \implies y^3 - 3y = 1 \qquad (b = -3, c = 1)$$

   which is the standard cubic with $b = -3$, $c = 1$. Cardano would have applied his modified algorithm for $y^3 = 3y + 1$, but we'll just using negative numbers in the original:

$$\begin{aligned} &\begin{cases} u^3 - v^3 = 1 \\ uv = -\frac{3}{3} = -1 \end{cases} \implies (u^3 + v^3)^2 = 1^2 + 4(-1)^3 = -3 \\ &\implies u^3 + v^3 = \sqrt{3}i \\ &\implies u^3 = \frac{1}{2}(\sqrt{3}i + 1), \quad v^3 = \frac{1}{2}(\sqrt{3}i - 1) \\ &\implies x = y - 1 = u - v - 1 = \sqrt[3]{\frac{\sqrt{3}i+1}{2}} - \sqrt[3]{\frac{\sqrt{3}i-1}{2}} - 1 \end{aligned}$$

   This is ugly! If you know Euler's formula and you choose a compatible pair of cube roots (remember that $uv = -1$), this will evaluate to one of three real numbers: the positive solution is in fact $x = 2\cos 20° - 1 \approx 0.8794$.

Factorization & the Fundamental Theorem of Algebra

By the late 1500s, Viète's abstraction allowed him to streamline Cardano's methods, as well as to investigate the relationship between the coefficients of a polynomial and its roots. For Viète the roots had to be positive, but later improvements by Thomas Harriot and Albert Girard (1629) applied this to all polynomials with any roots.

Here is the approach for the quadratic $ax^2 + bx + c = 0$, with roots r_1, r_2:

$$\begin{aligned}(r_1 - r_2)\big(a(r_1 + r_2) + b\big) &= a(r_1^2 - r_2^2) + b(r_1 - r_2) \\ &= \big(ar_1^2 + br_1 + c\big) - \big(ar_2^2 + br_2 + c\big) = 0\end{aligned}$$

Provided the roots are distinct,[50] we conclude that

$$\frac{b}{a} = -(r_1 + r_2), \quad \text{and} \quad \frac{c}{a} = -r_1^2 - \frac{b}{a}r_1 = -r_1^2 + (r_1 + r_2)r_1 = r_1 r_2$$

These are the quadratic version of what are now known as *Viète's formulas*; versions exist for polynomials of every degree. Their use amounts to an early form of factorization.

Example By inspection, we spot that $3x^2 - 2x - 1 = 0$ has $r_1 = 1$ as a root. Applying Viète's formulas,

$$r_1 + r_2 = -\frac{b}{a} = \frac{2}{3} \implies r_2 = -\frac{1}{3} \quad \text{or alternatively} \quad r_1 r_2 = \frac{c}{a} = -\frac{1}{3} \implies r_2 = -\frac{1}{3}$$

Think about the relationship between this approach and factorization!

A nice side-effect is a method for obtaining the quadratic formula by solving a pair of simultaneous equations analogous to Cardano's cubic approach:

$$\begin{cases} r_1 + r_2 = -\frac{b}{a} \\ r_1 - r_2 = \sqrt{(r_1 + r_2)^2 - 4r_1 r_2} = \sqrt{\left(\frac{b}{a}\right)^2 - 4\frac{c}{a}} \end{cases} \implies r_1, r_2 = -\frac{b}{2a} \pm \frac{1}{2}\sqrt{\left(\frac{b}{a}\right)^2 - 4\frac{c}{a}}$$

Viète's formulas were central to the later development of Galois Theory (1830) and the Abel–Ruffini theorem regarding the insolubility of quintic and higher-degree polynomials.

The remaining key results regarding solutions of polynomials also appeared around this time:

Fundamental Theorem of Algebra Including multiplicity, a degree n polynomial has n complex roots. Girard offered the first version in 1629, though a complete proof didn't appear until the work of Argand, Cauchy and Gauss in the early 1800s.

Factor Theorem In 1637, Descartes proved a familiar result, essentially via long-division:

$$p(r) = 0 \iff p(x) \text{ is divisible by } x - r.$$

[50] In fact the formulas hold even when $r_1 = r_2$.

For instance, here is Descartes' method to find the roots of $p(x) = x^3 + 2x^2 - 13x + 10$:

- Observe that $p(1) = 0$. Thus $x - 1$ is a factor.
- Use long-division to obtain $p(x) = (x-1)(x^2 + 3x - 10)$.
- $q(x) = x^2 + 3x - 10$ has $q(2) = 0$; divide to get $q(x) = (x-2)(x+5)$.
- The roots of $p(x)$ are therefore $1, 2$ and -5 (Descartes called this last a *false root*).

We'll return to Descartes in the context of Analytic Geometry in the next chapter.

Exercises 7.2. *Key concepts: Cardano's method for $x^3 + bx = c$, Abstraction via arbitrary constants, Investigating the relationship between coefficients and roots*

1. (a) Find Viète's formulas for the polynomial $p(x) = x^3 + ax^2 + bx + c$ with roots r_1, r_2, r_3; that is, find the coefficients a, b, c in terms of the roots.
   (*Hint: Multiply out $p(x) = (x - r_1)(x - r_2)(x - r_3)\ldots$*)

   (b) Solve $x^3 - 6x^2 + 9x - 4 = 0$ using Girard's method: first, determine one solution by inspection, then use Viète's formulas for the cubic to investigate the relationship between the remaining roots.

2. (a) Apply Cardano's method to the equation $x^3 + 6x = 20$.
   (*Hint: to finish, compute $(1+\sqrt{3})^3$*)

   (b) If $b, c > 0$, Cardano's method finds a single positive solution to $x^3 + bx = c$. Explain why such an equation always has exactly one real solution which is moreover positive.

3. Prove that if t is a root of $x^3 = cx + d$, then

$$r_1 = \frac{t}{2} + \sqrt{c - \frac{3t^2}{4}} \quad \text{and} \quad r_2 = \frac{t}{2} - \sqrt{c - \frac{3t^2}{4}}$$

   are both roots of $x^3 + d = cx$. Use this to solve $x^3 + 3 = 8x$.

4. Consider the cubic equation $aaa - 3raa + ppa = 2xxx$ (as written by Harriot). Show that the substitution $a = e + r$ reduces this to an equation without a square term.

   As an example, reduce the equation $aaa - 18aa + 87a = 110$ to a cubic in e without a square term. Find all three solutions (in e) and thus the solutions to the original equation (in a).

5. Find all roots of the cubic $p(x) = 2x^3 - 3x^2 - 3x + 2$ using Descartes' factor theorem.

6. Use Cardano's method (with Tschirnhaus substitution and complex numbers!) to find the solutions to the equation $x^3 + 4 = 3x^2$. Verify that you get the same solutions using Descartes' factor theorem.

   (*Hint: all solutions are integers!*)

7. (If you are very comfortable with complex numbers) Use Euler's formula to verify that the equation $x^3 + 3x^2 = 3$ has the positive solution $x = 2\cos 20^\circ - 1$. It also has two negative real solutions: find them!