# 4 Ancient Astronomy & Trigonometry

Throughout history, astronomy has been (and remains) an important driver of mathematical development. In particular, early *trigonometry* (*triangle-measure*) was largely developed to facilitate astronomical computations, for which there are many practical benefits: for instance,

*Calendars* The phases of the moon (whence *month*), the seasons, and the solar year are paramount. Without an accurate calendar, food production, gathering and hunting are more difficult: *When* will the rains come? *When* should we plant/harvest? *When* will the buffalo return?

*Navigation* The simplest navigational observation in the northern hemisphere is that the stars appear to orbit *Polaris* (the pole star), thus providing a fixed reference point/direction in the night sky. As humans travelled further, accurate computations became increasingly important.

**Religion and Astrology** In modern times, we distinguish *astronomy* (the science) from *astrology* (how the heavens influence our lives). However, for most of human history the two were inseparable. In light-polluted modern cities, it is hard to imagine the significance the night sky held for our ancestors, even a couple of centuries ago. Almost all religions imbue the heavens with meaning; understanding and predicting heavenly movements provided a massive historically imperative for mathematical and technological development. Here are just a few examples of the relationship between astronomy, astrology and culture.

- The concept of *heaven* as the domain of the gods, whether explicitly in the sky or simply atop a high mountain (e.g., Olympus in Greek mythology, Moses ascending Mt. Sinai, etc.).
- Many ancient structures were constructed in alignment with heavenly objects:
  - Ancient Egyptians viewed the region around Polaris as their heaven; pyramids included shafts emanating from the burial chamber so that the deceased could 'ascend to the stars.'
  - Several Mayan temples and observatories appear to be oriented to the solstices (page 37). Such alignments are also found elsewhere in the Americas and throughout the world.
  - Venus and Sirius—respectively the brightest planet and star in the night sky—were also important objects of alignment.
- The modern (western) zodiac comes from pre-1000 BCBabylon. A tablet dated to 686 BC describes 60–70 constellations and stars with aspects familiar to modern astrologers, including Taurus, Leo, Scorpio and Capricorn. During the same period Chinese and Indian astronomers developed different systems of constellations.[16]
- Calendars mark religious festivals, practices and even the age of the world.
  - The traditional Hebrew calendar dates the beginning of the world to 3760 BC.
  - The Mayan long count calendar dates the creation of the world to 3114 BC.
  - The modern Gregorian calendar arose to facilitate an accurate determination of Easter.
- The *star in the east* is associated to the birth of Jesus in Christianity.
- Muslims orient themselves towards Mecca when at prayer; we'll see later how this direction (the *qibla*) may be computed, but the required data is astronomical.

[16]Chinese astronomy has 28 constellations (or *mansions*). As a point of comparison, Taurus corresponds roughly to the Chinese 'White Tiger of the West' (*Baihu*, and similar terms in various East-Asian languages).

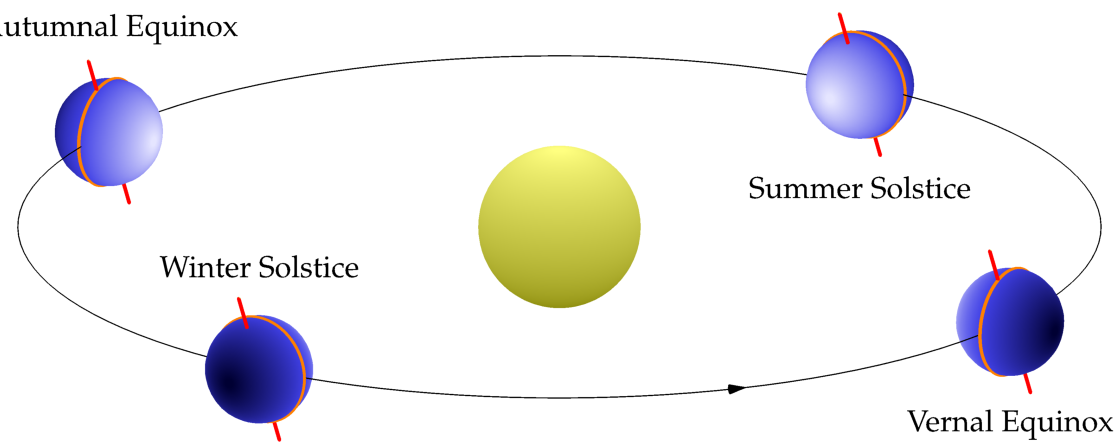
## 4.1 Astronomical Terminology and Early Measurement

Seasonal variation exists because the earth's axis is tilted approximately 23.5° with respect to the *ecliptic* (sun-earth orbital plane). Summer, in a given hemisphere, is when the earth's axis is tilted towards the sun, resulting in more sunlight and longer days. Astronomically, the seasons are determined by four dates:

*Summer/Winter Solstices* ($\approx 21^{\text{st}}$ June/December) The north pole is maximally tilted towards/away from the sun. *Solstice* comes from Latin meaning 'sun stationary.' The location of the rising/setting sun and its maximal (noon) elevation changes throughout the year, with extremes on the solstices, the summer solstice being when the setting sun is most northerly and (north of the tropics) the noon sun is highest in the sky. Indeed the tropics (of Cancer/Capricorn) are the lines of latitude where the sun is directly overhead at noon on one solstice.

*Vernal/Autumnal Equinoxes* ($\approx 21^{\text{st}}$ March/September) Earth's axis is perpendicular to the Sun-Earth orbital radius. *Equinox* means *equal night*: day and night both last approximately 12 hours everywhere since Earth's axis passes through the day-night boundary.

The picture shows the orientation of the ecliptic, the earth's axis and the day-night boundary.




Astronomical measurements must be be conducted relative to this set-up.

*Fixed stars* These form the background with respect to which everything else is measured. The *ecliptic* is the sun's apparent path over the year set against the fixed stars. Planets (*wandering stars*) are also seen to move relative to this background.

*Celestial longitude* Measured from zero to 360° around the ecliptic with 0° at the vernal equinox. One degree corresponds approximately to the sun's apparent daily motion. The ecliptic is divided into twelve equal segments: Aries is 0–30° (March to April $21^{\text{st}}$); Taurus is 30–60°, etc.

*Celestial latitude* Measured in degrees north or south of the ecliptic; the sun has latitude zero.

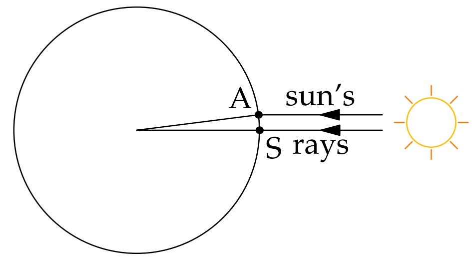
This formulation was largely co-opted by the Greeks from Babylon. The Greeks kept the Babylonian base-60 degrees-minutes-seconds system, which, with minor modifications, persists to this day.[17]

[17] Modern astronomers typically measure latitude and longitude (declination/right-ascension) with respect to the earth's equatorial plane rather than the ecliptic. Such *equatorial co-ordinates* are first known to have been introduced by Hipparchus of Nicaea (page 40). Right-ascension is measured in hours-minutes-seconds rather than degrees, where 24 hours = 360°, though modern scientific practice is to use decimals rather than sexagesimal minutes and seconds.

**The Circumference of the Earth**

One of the earliest problems in astronomy is to find the size of the earth. Eratosthenes of Cyrene (c. 200 BC, page 34) performed one of the first accurate estimations by measuring the sun's rays at noon in two different places.

- Syene (modern-day Aswan, Egypt) is approximately 5,000 *stadia* south of Alexandria.
- When the sun is directly overhead at Syene, it is inclined $7°12' = \frac{1}{50} \cdot 360°$ at Alexandria.
- The circumference of the earth is therefore approximately $50 \cdot 5000 = 250,000$ stadia.




Eratosthenes' original calculation is lost, though it was a little more complicated than the above. From other (shorter) distances, historians have inferred that Eratosthenes' *stadion* was $\approx 172$ yards, making his approximation for the circumference of the earth $\approx 24,500$ miles, astonishingly accurate in comparison to the modern value of $\approx 25,000$ miles. Later mathematicians provided other estimates based on other locations, but the basic method was the same.

**Modelling the Heavens**

Early Greek analysis reflects several assumptions.

- Spheres and circles are perfect, matching the 'perfect design' of the universe. The earth is a sphere and the fixed stars (constellations) lie on a larger 'celestial sphere.' Models relied on spheres and circles rotating at constant rates.
- The earth is stationary, and the celestial sphere rotates around it once per day.
- The planets lie on concentric spherical shells centered on the earth.

When such assumptions are tested by observation, two major contradictions appear:

*Variable brightness* The apparent brightness of heavenly bodies, particularly planets, is non-constant.

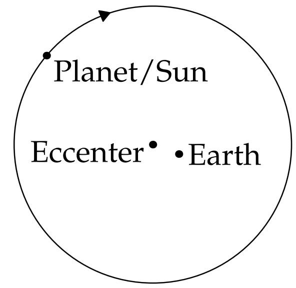
*Retrograde motion* Planets mostly follow the east-west motion of the heavens, though sometimes they are seen to slow down and reverse course.

If planets move at constant speed around circles centered on the earth, how can these observations be explained? Attempts to produce accurate models while preserving spherical/circular motion led to the development of new mathematics.

One early approach is due to Eudoxus of Knidos (c. 370 BC, page 21), who developed a concentric-sphere model where planets and the sun are attached to separate spheres, each of which has its poles attached to the sphere outside it, with the outermost sphere being that of the fixed stars. The motion generated by such a model[18] is highly complex. Eudoxus' approach is capable of producing retrograde motion, but not the variable brightness of stars and planets.

[18]The link is to a very nice flash animation of Eudoxus' model that would have been far beyond Eudoxus' ability to visualize and measure.

**Epicycles & Eccentric Orbits** Apollonius of Perga (2nd/3rd C. BC) is most famous for his study of conic sections, but is relevant here for developing two models of solar/planetary motion.



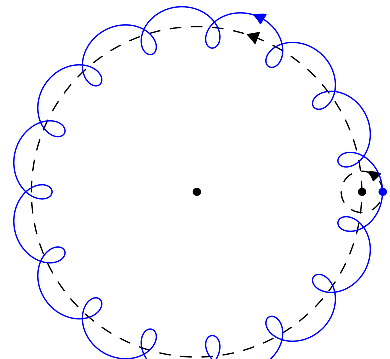

In the *eccenter* model, a planetary/solar orbit is a circle (the deferent) whose center is *not* the earth. This straightforwardly addresses the problem of variable brightness since the planet is not a fixed distance from the earth.

The obvious criticism is *why?* What philosophical justification could there be for the eccenter? Eudoxus' model might have been complex and impractical, but was more in line with the assumptions of spherical/circular motion.

Apollonius' second approach used *epicycles*: small circles attached to a larger circle—you'll be familiar with these if you've played with the toy *Spirograph*. An observer at the center sees the apparent brightness change, and potentially observes retrograde motion. In modern language, the motion is parametrized by the vector-valued function

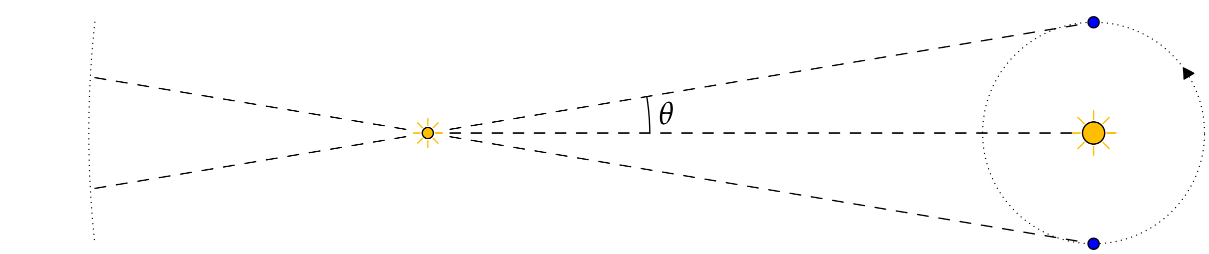
$$\mathbf{x}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} + r\begin{pmatrix}\cos\psi t\\ \sin\psi t\end{pmatrix}$$

where $R, r, \omega, \psi$ are the radii and frequencies (rad/s) of the circles.



Combining these models allowed Apollonius to describe very complex motion. Calculation was difficult however, requiring finding lengths of chords of various circles from a given angle, and vice versa. It is from this requirement that some of the earliest notions of trigonometry arose.

One might ask why the Greeks didn't make the 'obvious' fix and place the sun at the center of the cosmos. In fact Aristarchus of Samos (c. 310–230 BC) did precisely this, suggesting that the fixed stars were really just other suns at exceptional distance! However, the great thinkers of the time (Plato, Aristotle, etc.) had a very strong objection to Aristarchus' proposal: *parallax.*

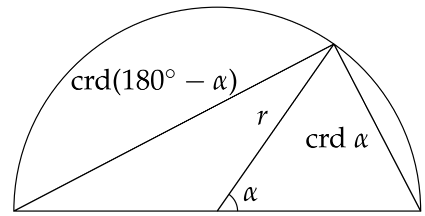



If the earth moves around the sun and the fixed stars are really independent objects, then the position of a nearer star should appear to change throughout the year. The angle $\theta$ in the picture is the *parallax* of the nearer star. Unfortunately for Aristarchus, the Greeks were incapable of observing any parallax.[19] It took 2000 years before the work of Copernicus and Kepler in the 15-1600s forced astronomers to take *heliocentric* models seriously (*Helios* is the Greek sun-god).

[19] The astronomical unit of one *parsec* is the distance to a star exhibiting one arc-second ($\frac{1}{3600}{}^\circ$) of parallax: roughly 3.3 light-years or $3 \times 10^{13}$ km, an unimaginable distance to anyone before the scientific revolution. Since the nearest star to our sun, *Proxima Centauri*, lies 4.2 light years = 0.77 parsecs away, the rejection of Aristarchus' hypothesis is understandable!

**Hipparchus of Nicaea/Rhodes (c. 190–120 BC)**

Born in Nicaea (northern Turkey) but doing much of his work on the Mediterranean island of Rhodes, Hipparchus was one of the pre-eminent Greek astronomers. He made use of Babylonian eclipse data to fit Apollonius' eccenter and epicycle models to the observed motion of the moon. As part of this work, he needed to accurately compute chords of circles. His *chord tables* are acknowledged as the earliest lists of trigonometric values.




In an imitation of Hipparchus' approach, we define a function crd which returns the length of the chord in a given circle subtended by a given angle. In modern language

$$\operatorname{crd}\alpha = 2r\sin\frac{\alpha}{2}$$

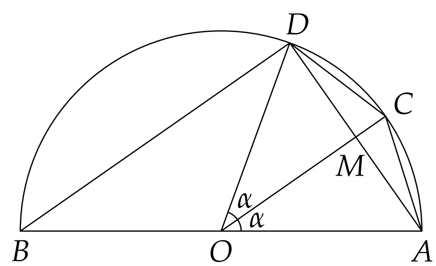
Hipparchus chose a circle with circumference 360° (in fact he used $60 \cdot 360 = 21600$ arc-minutes), whence $r = \frac{21600}{2\pi} \approx 57, 18;$ (base-60). Note that this is sixty times the number of *degrees per radian*.[20] His chord table was constructed starting with two obvious values:

$$\operatorname{crd}60^\circ = r = 57, 18; \qquad \operatorname{crd}90^\circ = \sqrt{2}r = 81, 2;$$

Since (Thales Theorem) the large triangle is right-angled, the Pythagorean theorem can be used to obtain chords for angles $180^\circ - \alpha$. In modern language

$$\operatorname{crd}(180^\circ - \alpha) = \sqrt{(2r)^2 - (\operatorname{crd}\alpha)^2} = 2r\sqrt{1-\sin^2(\alpha/2)} = 2r\cos\frac{\alpha}{2}$$

Pythagoras was again used to halve and double angles in an approach analogous to Archimedes' quadrature of the circle (page 33). We rewrite the argument in this language.




In the picture, we double the angle $\alpha$; plainly $M$ is the midpoint of $\overline{AD}$ and $|DB| = \operatorname{crd}(180^\circ - 2\alpha)$. Since $\angle BDA = 90^\circ$, it follows that $\overline{BD}$ is parallel to $\overline{OM}$ and so

$$|OM| = \frac{1}{2}|BD| = \frac{1}{2}\operatorname{crd}(180^\circ - 2\alpha)$$

Now apply Pythagoras to $\triangle CMD$:

$$\begin{aligned}
(\operatorname{crd}\alpha)^2 &= \left(\frac{1}{2}\operatorname{crd}2\alpha\right)^2 + \left(r - \frac{1}{2}\operatorname{crd}(180^\circ - 2\alpha)\right)^2 && (|CD|^2 = |DM|^2 + |CM|^2)\\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180^\circ - 2\alpha) + \frac{1}{4}\operatorname{crd}(180^\circ - 2\alpha)^2\\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180^\circ - 2\alpha) + \frac{1}{4}(4r^2 - (\operatorname{crd}2\alpha)^2)\\
&= 2r^2 - r\operatorname{crd}(180^\circ - 2\alpha) = 2r^2 - r\sqrt{4r^2 - (\operatorname{crd}2\alpha)^2}
\end{aligned}$$

In modern notation this is one of the double-angle trigonometric identities!

$$4r^2\sin^2\frac{\alpha}{2} = 2r^2 - 2r^2\cos\alpha \iff \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

[20] One radian is the angle subtended by an arc equal in length to the radius of the circle. Hipparchus essentially does this in reverse: the circumference is fixed so that *degree* now measures both subtended angle *and* circumferential distance.

**Example** To calculate crd 30°, we start with crd 60° $= r$. Then

$$\begin{aligned} &\operatorname{crd} 120^\circ = \sqrt{4r^2 - r^2} = \sqrt{3}\, r \\ \implies &\operatorname{crd} 30^\circ = \sqrt{2r^2 - r \operatorname{crd}(180^\circ - 60^\circ)} = \sqrt{2r^2 - \sqrt{3}r^2} = \sqrt{2 - \sqrt{3}}\, r \end{aligned}$$

In modern language this yields an exact value for sin 15°:

$$\operatorname{crd} 30^\circ = 2r \sin 15^\circ \implies \sin 15^\circ = \frac{1}{2}\sqrt{2 - \sqrt{3}}$$

Continuing this process, we obtain crd 150° $= \sqrt{2 + \sqrt{3}}\, r$, whence

$$(\operatorname{crd} 15^\circ)^2 = 2r^2 - r \operatorname{crd} 150^\circ = \left(2 - \sqrt{2 + \sqrt{3}}\right) r^2 \implies \operatorname{crd} 15^\circ = \sqrt{2 - \sqrt{2 + \sqrt{3}}}\, r$$

Again translating: $\sin 7.5^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{3}}}$.
In similar fashion, Hipparchus computed the chords of 7.5°, 15°, ..., 172.5°, in steps of 7.5°. Of course everything was an estimate since he had to rely on repeated approximations for square-roots. All Hipparchus' original work is lost, so we know of his approach only by reference. In particular, while the above method is probably due to Hipparchus, we see it first in the work of Ptolemy, as we'll consider next.

**Exercises 4.1.** 1. Calculate crd 150°, crd 165°, and crd $172\frac{1}{2}°$ using the method of Hipparchus.
(*Leave your answers as a multiple of* $r =$ crd 60°)

2. *Sirius*, the brightest star in the sky, is 2.64 parsecs (8.6 light-years) from the sun. Use modern trigonometry to find its parallax.

3. The tropic of cancer is the line of latitude (approximately) 23.5° north of the equator marking the locations where the sun is directly overhead at noon on the summer solstice.[21] At the arctic circle on the *winter* solstice, the sun is precisely on the horizon.
   (a) Explain why the latitude of the arctic circle is 66.5° north.
   (b) Find the angle the sun makes *above* the horizon at the arctic circle at noon on the summer solstice.

4. Consider the epicycle model where the position vector of a planet is given by

$$\mathbf{x}(t) = R \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} + r \begin{pmatrix} \cos \psi t \\ \sin \psi t \end{pmatrix}$$

   (a) Suppose $R = 4$ and $r = 1$, $\omega = 1$ and $\psi = 2$, so that the epicycle rotates twice every orbit. Sketch a picture of the full orbit.
   (b) Suppose that $\omega, \psi$ are positive constants. Prove that an observer will see retrograde motion if and only if $r\psi > R\omega$.
   (*Hint: differentiate* $\mathbf{x}'(t)$ *and think about its direction*)

[21] Syene (page 38) is almost exactly on the Tropic of Cancer.

## 4.2 Ptolemy's *Almagest*

Born in Egypt and living much of his life in Alexandria, Claudius Ptolemy (c. AD 100–170) was a Greek/Egyptian/Roman[22] astronomer and mathematician. Around AD 150, he produced the *Mathematica Syntaxis*, better known as the *Almagest*. The latter term is derived from the Arabic *al-mageisti* (great work), reflecting its importance to later Islamic learning.

The *Almagest* is essentially a textbook on geocentric cosmology. It shows how to compute the motions of the moon, sun and planets, describing lunar parallax, eclipses, the constellations, and elementary spherical trigonometry (this last probably courtesy of Menelaus c. AD 100). It contains our best evidence as to the accomplishments of Hipparchus and describes his calculations. The *Almagest* formed the basis of Western/Islamic astronomical theory well into the 1600s.

**Ptolemy's Calculations** Ptolemy used several innovations to compute more chords at greater accuracy than Hipparchus.

*Initial Data* Ptolemy took $r = 60$ so that $\operatorname{crd} 60° = 60$. He also had more initial data:

$$\operatorname{crd} 90° = 60\sqrt{2}, \quad \operatorname{crd} 36° = 30(\sqrt{5}-1), \quad \operatorname{crd} 72° = 30\sqrt{10-2\sqrt{5}}$$

*Halving/Doubling Angles* Ptolemy used what was probably Hipparchus' method:

$$\operatorname{crd}^2 \alpha = 2r^2 - r\operatorname{crd}(180° - 2\alpha) = 60\big(120 - \operatorname{crd}(180° - 2\alpha)\big)$$
$$\operatorname{crd}(180° - \alpha) = \sqrt{(2r)^2 - \operatorname{crd}^2\alpha} = \sqrt{120^2 - \operatorname{crd}^2\alpha}$$

approximating square-roots to the desired accuracy. For example,

$$\operatorname{crd} 30° = \sqrt{60(120 - \operatorname{crd} 120°)} = \sqrt{60\left(120 - 60\sqrt{3}\right)} = 60\sqrt{2-\sqrt{3}} \approx 31;3,30$$

*Multiple-Angle Formula* Ptolemy computed $\operatorname{crd} 12° = \operatorname{crd}(72° - 60°)$, then halved this for angles of 6°, 3°, 1.5°, and 0.75°. Chords for all integer multiples of 1.5° were computed using multiple-angle/addition formulæ.

*Interpolation* The observation $\alpha < \beta \implies \frac{\operatorname{crd}\beta}{\operatorname{crd}\alpha} < \frac{\beta}{\alpha}$ allowed Ptolemy to compute chords for every half-degree to the incredible accuracy of two sexagesimal places. For approximating between half-degrees, his table indicated how much should be added for each arc-minute ($\frac{1}{60}°$). For example, the second line of Ptolemy's table reads

$$1° \qquad 1;2,50 \qquad ;1,2,50$$

The first two columns state that $\operatorname{crd} 1° = 1;2,50$ to two sexagesimal places.[23] The third entry says, for example, that

$$\operatorname{crd} 1°5' \approx 1;2,50 + 5(;1,2,50) = 1;8,4,10 \approx 1;8,4$$

To obtain these arc-minute approximations, it is believed Ptolemy computed half-angle chords to an accuracy of *five sexagesimal places* (1 part in over 750 million!). The construction of the chord-table must have been a gargantuan task, one for which Ptolemy likely had much assistance.

[22] Ptolemy (Ptolemaeus) is a Greek name, while Claudius is Roman, reflecting the changing cultural situation in Egypt.

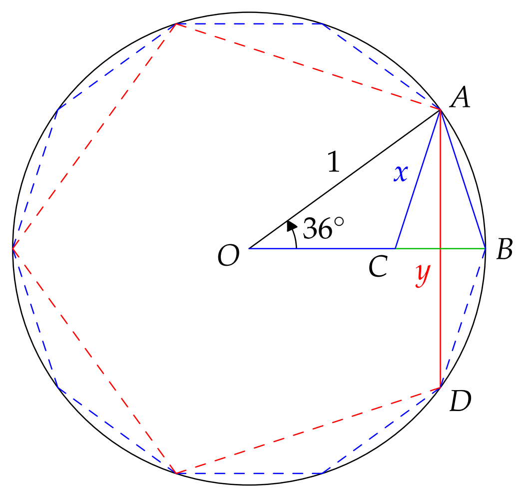
[23] This is $1 + \frac{2}{60} + \frac{50}{60^2} = 1.0472222\ldots = 120\sin\frac{1.00003625\ldots}{2}°$, an already phenomenal level of accuracy.

**How did Ptolemy know exact values for** crd 36° **and** crd 72°**?** Everything is in Euclid's *Elements*!

**Theorem.** 1. *(Thm XIII. 9) In a circle, the sides of a regular inscribed hexagon and decagon are in the golden ratio (this ratio is* $60 : \text{crd}\,36°$ *in Ptolemy).*

2. *(Thm XIII. 10) In a circle, the square on an inscribed pentagon equals the sum of the squares on an inscribed hexagon and decagon.*

Purely Euclidean proofs are too difficult for us, so here is a way to see things in modern notation.




1. Let $\overline{AB} = x$ be the side of a regular decagon inscribed in a unit circle with center $O$.

   $\triangle OAB$ is isosceles with angles $36°, 72°, 72°$.

   Let $C$ lie on $\overline{OB}$ such that $\overline{AC} = x$.

   Count angles to see that $\triangle OAB$ and $\triangle ABC$ are similar, that $\angle OAC = 36°$ and so $\overline{OC} = x$.

   Similarity now tells us that

   $$x = \frac{1-x}{x} \implies x = \frac{\sqrt{5}-1}{2}$$

   In a circle of radius 60, this gives the exact value

   $$\text{crd}\,36° = 60x = 30(\sqrt{5}-1)$$

2. Now let $\overline{AD} = y$ be the side of a regular pentagon inscribed in the same circle. Applying Pythagoras, we see that

   $$\left(\frac{y}{2}\right)^2 + \left(\frac{1-x}{2}\right)^2 = x^2$$

   Since $x^2 = 1 - x$, this multiplies out to give Euclid's result

   $$y^2 = 1^2 + x^2$$

   from which we obtain the exact value

   $$\text{crd}\,72° = 60y = 30\sqrt{10-2\sqrt{5}}$$

While these values were geometrically precise, Ptolemy used sexagesimal approximations to square-roots to obtain the values stated in his tables:

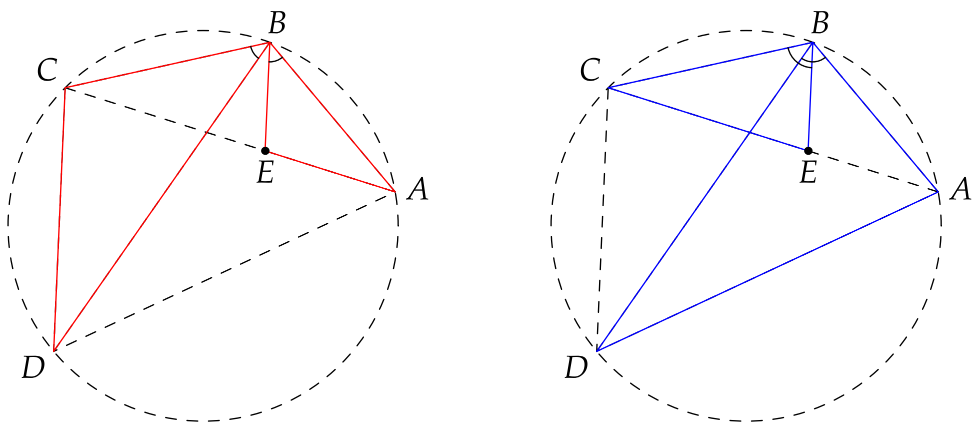
$$\text{crd}\,36° = 37;4,55 \qquad \text{crd}\,72° = 70;32,3$$

He must have used a far higher degree of accuracy in order to obtain similarly accurate values for other chords.

**Angle-addition & the Multiple-angle Formula** Computation of $\operatorname{crd}(\alpha \pm \beta)$ was facilitated by versions of the multiple-angle formulæ of modern trigonometry.

**Theorem (Ptolemy's Theorem).** *Suppose a quadrilateral is inscribed in a circle. Then the product of the diagonals equals the sum of the products of the opposite sides.*[24]

*Proof.* Choose $E$ on $\overline{AC}$ such that $\angle ABE \cong \angle DBC$. Then $\angle ABD \cong \angle EBC$. Since $\angle BAE \cong \angle BDC$ are inscribed angles of the same arc $\overline{BC}$, we obtain two pairs of similar triangles:




$\triangle ABE \sim \triangle DBC$ $\triangle ABD \sim \triangle EBC$

The proof follows immediately: since $\frac{|AE|}{|CD|} = \frac{|AB|}{|BD|}$ and $\frac{|CE|}{|AD|} = \frac{|BC|}{|BD|}$, we have

$$|AC|\,|BD| = (|AE| + |CE|)\,|BD| = |AB|\,|CD| + |AD|\,|BC|$$

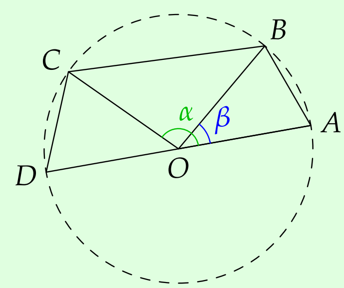
■

**Corollary.** *If $\alpha > \beta$, then*

$$120 \operatorname{crd}(\alpha - \beta) = \operatorname{crd}\alpha \operatorname{crd}(180° - \beta) - \operatorname{crd}\beta \operatorname{crd}(180° - \alpha)$$

*In modern language, divide through by $120^2$ to obtain*

$$\sin\frac{\alpha-\beta}{2} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\alpha}{2}$$




*Proof.* If $|AD| = 120$ is a diameter of the pictured circle, then Ptolemy's Theorem says

$$\operatorname{crd}\alpha \operatorname{crd}(180° - \beta) = \operatorname{crd}\beta \operatorname{crd}(180° - \alpha) + 120 \operatorname{crd}(\alpha - \beta)$$

■

Similar expressions for $\operatorname{crd}(\alpha + \beta)$ and $\operatorname{crd}(180° - (\alpha \pm \beta))$ were also obtained, essentially recovering all versions of the modern multiple-angle formulæ for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$.

[24]It is generally considered that this result predates Ptolemy, though there is some debate as to whether it belongs in the *Elements*. Book VI traditionally contains 33 propositions, however some editions append four corollaries of which Ptolemy's Theorem is the last (Thm VI. D).

**Examples** 1. Here is how Ptolemy might have calculated $\operatorname{crd} 42°$. Let $\alpha = 72°$ and $\beta = 30°$, then

$$120 \operatorname{crd} 42° = \operatorname{crd} 72° \operatorname{crd} 150° - \operatorname{crd} 30° \operatorname{crd} 108°$$

Since $\operatorname{crd} 72° = 30\sqrt{10 - 2\sqrt{5}}$ is known, and

$$\operatorname{crd} 108° = \operatorname{crd}(180° - 2 \cdot 36°) = 120 - \frac{1}{60}\operatorname{crd}^2 36° = 30(1+\sqrt{5})$$

we see that

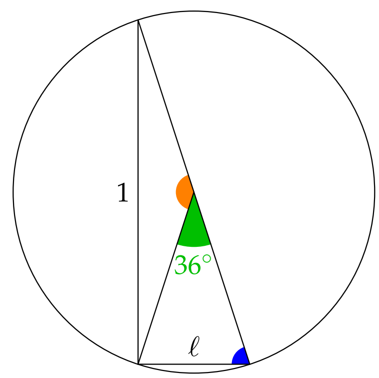
$$\begin{aligned}\operatorname{crd} 42° &= \frac{1}{120}\left(30\sqrt{10-2\sqrt{5}} \cdot 60\sqrt{2+\sqrt{3}} - 60\sqrt{2-\sqrt{3}} \cdot 30(1+\sqrt{5})\right) \\ &= 15\left(\sqrt{10-2\sqrt{5}} \cdot \sqrt{2+\sqrt{3}} - (1+\sqrt{5})\sqrt{2-\sqrt{3}}\right) \approx 43;0,15 \approx 43.0042\end{aligned}$$

Note all the square-roots which had to be approximated!

2. The *Almagest* also contained many practical examples. Here is one such.

A stick of length 1 is placed in the ground. The angle of elevation of the sun is 72°. What is the length of its shadow?

Ptolemy tells us to draw a picture. The lower isosceles triangle has base angles 72° and the length of the shadow is $\ell$. The ratio of the chords is then computed:




$$1 : \ell = \operatorname{crd} 144° : \operatorname{crd} 36°$$

$$\implies \ell = \frac{\operatorname{crd} 36°}{\operatorname{crd} 144°} = \frac{30(\sqrt{5}-1)}{30\sqrt{10+2\sqrt{5}}} \approx 0.32491$$

This is precisely $\cot 72°$, though Ptolemy had no such notion.

**Exercises 4.2.** 1. What are the exact values of $\sin 36°$ and $\sin 18°$?

2. (a) Restate the interpolation formula $\alpha < \beta \implies \frac{\operatorname{crd}\beta}{\operatorname{crd}\alpha} < \frac{\beta}{\alpha}$ in terms of the sine function. What facts about $\frac{\sin x}{x}$ does this reflect?
   (b) Find $\operatorname{crd} 57'$ (arc-minutes!) to two sexagesimal places.

3. Find the exact value of $\operatorname{crd} 54°$

4. Prove the following using Ptolemy's Theorem. What is this in modern language?

$$120 \operatorname{crd}\big(180° - (\alpha+\beta)\big) = \operatorname{crd}(180° - \alpha)\operatorname{crd}(180° - \beta) - \operatorname{crd}\alpha \operatorname{crd}\beta$$

5. Use Ptolemy's Theorem to establish a version of the double-angle formula: $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$.
   (*Hint: draw a symmetric quadrilateral one of whose diagonals is a diameter*)

6. Calculate the length of a noon shadow of a pole of length 60 using Ptolemy's methods:
   (a) On the vernal equinox at latitude 40°.
   (b) At latitude 36° north on both the summer and winter solstices.
   (*Hint: recall Exercise 4.1.3*)