3 Ancient Greek Mathematics

3.1 Overview of Ancient Greek Civilization & Early Philosophy

Ancient Greek civilization dates from around 800 BC, centered on the peninsula between the Adriatic sea (to the west) and the Aegean (to the east) that forms the mainland of the modern Greek state. Ancient Greek culture was decentralized, consisting of semi-independent city-states connected by trade. Accomplished sea-farers, they extended their reach round the northern and eastern coasts of the Mediterranean, from Iberia (Spain) to the Black Sea, Anatolia (western Turkey), and Egypt, building and capturing a network of city-states and trading outposts.

Philip of Macedon 'unified' the Greek peninsula just before his death in 336 BC. His son, Alexander 'the Great,' launched a massive campaign conquering Persia, Egypt, Babylon, and western India before his own death in Babylon in 323 BC. Alexander left provincial governors to manage captured territory; some of these structures endured for centuries (Egypt's Ptolemaic dynasty), whereas others were overthrown after only a few years (parts of India). While Alexander's conquests did not produce a long-lasting centralized Greek empire, they were effective at expanding the reach of Greek cultural practice and philosophy, and brought external ideas into the Greek tradition.

The core part of Greek territory was absorbed by the Roman empire around 146 BC. In line with typical Roman practice, those Greeks who agreed to accept Roman governors and taxation became Roman citizens.[6] As such, the Greek culture of inquiry and scholarship was left largely intact under Roman rule. Greek culture and learning was central to the later Byzantine (eastern Roman) empire (centered on Byzantium/Constantinople/Istanbul), which lasted until Constantinople was captured by the Islamic Ottomans in 1453. For centuries prior, Islamic scholarship had itself been significantly influenced by the ancient Greeks; the main consequence of the fall of Constantinople for knowledge transfer was to encourage the exodus of scholars to Rome, helping to fuel the nascent European Renaissance.

Greek Territory c. 500 BC

[6]While this might sound reasonable, resistance wasn't a realistic option. . .

Greek mathematics is part of a much wider development of science and philosophy encompassing a change of emphasis from practicality to abstraction. One reason for this was the Greek blending of religion/mysticism with natural philosophy: a desire to describe the natural world while preserving the perfection/logic in the gods' design.

Early Greek inquiry into natural phenomena was encouraged through the personification of nature (e.g., sky = man, earth = woman). By 600 BC, philosophers were attempting to describe such phenomena in terms of natural predictable causes and structures. For example, some viewed matter as being comprised of the 'four elements' (fire, earth, water, air) combined in the correct proportions. While the system of the world was seen as divinely-designed, explanations relying on the whims of the gods were discouraged.

While the Greeks certainly used mathematics for practical purposes, philosophers idealized logic and were unhappy with approximations. This led to the development of *axiomatics, theorems* and *proof*, concepts for which there is scant pre-Greek evidence. The ancient Greek language is indeed the source of three words of critical importance:

Mathematics From *mathematos* (μαθήματος), meaning knowledge or learning; the term covered essentially anything that might be taught in Greek schools.

Geometry Literally *earth-measure*, a combination of two terms:

- *Gi* (γη) Dates from pre-5th century BC, meaning *land, earth* or *soil.* Capitalized (Γη) it could refer to the *Earth* (as a goddess).
- *Metron* (μέτρον) A *weight* or *measure*, a *dimension* (length, width, etc.), or the *metre* (rhythm) in music.

Theorem From *theoreo* (θεώρέω), meaning 'I contemplate/consider.' In a mathematical context this become *theorema* (θεώρημα): a proposition to be proved.

Ancient Greece had several schools, mostly private and open only to men. Typically arithmetic was taught until age 14, followed by geometry and astronomy until age 18. The most famous scholars of ancient Greece were the Athenian trio of Socrates, Plato and Aristotle,[7] whose writings became central to the Western/Islamic philosophical tradition. Plato's *Academy* in Athens was a model for centuries of schooling; the centrality of geometry to the curriculum was evidenced by the famous inscription above its entryway: "Let none ignorant of geometry enter here."

Ancient Greek Enumeration

The Greeks had two primary forms of enumeration, both dating from around 800–500 BC.

In *Attic Greek* (Attica = Athens) strokes were used for 1–4, with larger numerals using the first letter of the words for 5, 10, 100, 1000 and 10000. For example,

- Πεντε (pente) is Greek for five, whence Π denoted the number 5.
- Δεκα (deca) means ten, so Δ represented 10.
- H (hekaton), X (khilias) and M (myrion/myriad) denoted 100, 1000 and 10000 respectively.
- Larger numbers were written using combinations of these symbols, similarly to both Egyptian hieroglyphs and (the later) Roman numerals: e.g., $\mathrm{XHH\Pi}|| = 1207$.

[7] Each taught his successor, with the birth of Socrates to the death of Aristotle covering 470–322 BC.

Ionic Greek (Ionia = mid Anatolian coast) numerals used the Greek alphabet, an approach possibly copied from Egyptian hieratic enumeration. Larger numbers used a left subscript mark (like a comma) to denote thousands and/or M (with superscripts) for 10000 as in Attic Greek. For example,

$$35298 = \text{͵λ͵εσϟη} = \overset{\gamma}{\text{M}}\text{͵εσϟη}$$

1	α	10	ι	100	ρ
2	β	20	κ	200	σ
3	γ	30	λ	300	τ
4	δ	40	μ	400	υ
5	ε	50	ν	500	ϕ
6	ϛ	60	ξ	600	χ
7	ζ	70	o	700	ψ
8	η	80	π	800	ω
9	θ	90	ϟ	900	ϡ

The ancient Greek alphabet included three archaic symbols ϛ ϟ ϡ (stigma, qoppa, sampi), with which you're likely unfamiliar.

The Greeks also used Egyptian fractions, denoting reciprocals with an accent over the symbol: e.g., $\acute{\vartheta} = \frac{1}{9}$. The use of Egyptian fractions dominated in Europe well into the middle ages.

Ionic Greek enumeration has persisted, with few changes, into modern times, although Hindu–Arabic numerals are also in common usage. Eventually a bar was placed over numbers to distinguish them from words (e.g., $\overline{\xi\vartheta} = 89$), while modern practice is to insert a *keraia* (similar to an apostrophe) at the end of a number: thus 35298 = ͵λ͵εσϟη′.

Both the Attic and Ionic systems were suitable for record-keeping but terrible for calculations! Later Greek mathematicians adapted the Babylonian sexagesimal system for calculation purposes, helping cement its modern use of in astronomy, navigation and time-keeping.

Exercises 3.1. 1. State the number 1789 in both Attic and Ionic notation.

2. Represent $\frac{8}{9}$ as a sum of distinct unit fractions (Egyptian style). Express the result in (Ionic) Greek notation.

   (*The answer to this problem is non-unique*)

3. For tax purposes, the ancient Greeks would approximate the area of a quadrilateral field by multiplying the averages of the two pairs of opposite sides. In one example, the two pairs of opposite sides were given as

$$a = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \quad \text{opposite} \quad c = \frac{1}{8} + \frac{1}{16}, \quad \text{and,}$$
$$b = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \quad \text{opposite} \quad d = 1$$

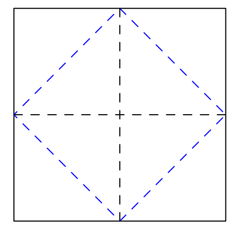
   where the lengths are in fractions of of a *schonion,* a measure of approximately 150 feet. Find the average of a and c, the average of b and d, and thus the approximate area of the field in square *schonion.* The taxman then rounds up the answer to collect a little more tax!

3.2 Pre-Euclidean Greek Mathematics

The publication of Euclid's *Elements* (c. 300 BC) forms a natural breakpoint in Greek mathematics, since is subsumed much of what came before it. In this section, we consider the contributions of several pre-Euclidean mathematicians. There are very few sources for Greek mathematics & philosophy before 400 BC, so almost everything is inferred from the writings and commentaries of others.[8]

Thales of Miletus (c. 624–546 BC) One of the first people known to state abstract general principles, Thales was a trader based in Miletus, a city-state in Anatolia. He travelled widely and was likely exposed to mathematical ideas from all round the Mediterranean. Here are some statements at least partly attributed to Thales:

- The angles at the base of an isosceles triangle are equal.
- Any circle is bisected by its diameter.
- A triangle inscribed in a semi-circle is right-angled (still known as Thales' Theorem).



Thales' major development is *generality*: his propositions concern *all* triangles, circles, etc. The Babylonians & Egyptians had examples of such results, but we have little indication that they viewed these abstractly. In accordance with the Greek idea of a theorem ('to look at'), Thales' reasoning was almost certainly pictorial. As an example of contemporary geometric reasoning, by 425 BC Socrates could describe how to halve/double the area of a square by joining the midpoints of edges.

Pythagoras of Samos (c. 572–497 BC) Like Thales, Pythagoras travelled widely, eventually settling in Croton (southeast Italy) where he founded a school lasting 100 years after his death. Plato is believed to have learned much of his mathematics from a Pythagorean named Archytas.

The Pythagoreans practiced a mini-religion whose ideas lay outside the mainstream of Greek society.[9] One of their mottos, "All is number," emphasised their belief in the centrality of pattern and proportion. The following quote[10] gives some flavor of the Pythagorean way of life.

> After a testing period and after rigorous selection, the initiates of this order were allowed to hear the voice of the Master [Pythagoras] behind a curtain; but only after some years, when their souls had been further purified by music and by living in purity in accordance with the regulations, were they allowed to see him. This purification and the initiation into the mysteries of harmony and of numbers would enable the soul to approach [become] the Divine and thus escape the circular chain of re-births.

The Pythagoreans were particularly interested in musical harmony and its relationship to number. They are credited with relating musical intervals to the ratios of lengths of vibrating strings:

- Identical strings whose lengths are in the ratio 2:1 vibrate an *octave* apart.
- A *perfect fifth* corresponds to the ratio 3:2.
- A *perfect fourth* corresponds to the ratio 4:3.

The use of these ratios to tune musical instruments is still known as *Pythagorean tuning.*

[8] For instance, most of our knowledge of Socrates comes from the voluminous writings of Plato and Aristotle. The earliest known Greek textbook/compilation (*Elements of Geometry*) was written around 430 BC by Hippocrates of Chios; no copy survives, though most of its material probably made it into Book I of Euclid.

[9] They were vegetarians, believed in the transmigration of souls, and accepted women as students; controversial indeed!

[10] Van der Waerden, *Science Awakening* pp 92–93

Theorems 21–34 in Book IX of Euclid's *Elements* are Pythagorean in origin. For instance:

Theorem. **(IX .21)** *A sum of even numbers is even.*
(IX .27) *Odd less odd is even.*

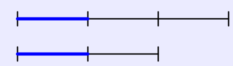
The Pythagoreans also studied perfect numbers, those which equal the sum of their proper divisors (e.g., $6 = 1 + 2 + 3$), and they seem to have observed the following famous result.

Theorem (IX. 36). *If $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is perfect.*

They moreover considered square and triangular numbers ($\frac{1}{2}m(m+1)$) and tried to express geometric shapes as numbers, in service of their belief that all matter could be formed from basic shapes.

Incommensurability and the Pythagorean Theorem As with other ancient cultures, the only numbers in Greek mathematics were *positive integers.* These were used to *compare* lengths/sizes of objects.

Definition. Lengths are in the ratio $m : n$ if some sub-length divides exactly m times into the first and n times into the second.
Two lengths are *commensurable* if some sub-length divides exactly into both.



Ratio 3 : 2

While modern mathematics has no problem with *irrational ratios* (e.g., the diagonal of a square to its side is $\sqrt{2} : 1$), this conflicted with the core Pythagorean belief that *any two lengths were commensurable.* Identifying lengths with real numbers (underlined), we restate their assertion in modern language:

$$\forall \underline{m}, \underline{n} \in \mathbb{R}^+, \ \exists \underline{\ell} \in \mathbb{R}^+, \ \exists a, b \in \mathbb{N}, \text{ such that } \underline{m} = a\underline{\ell} \text{ and } \underline{n} = b\underline{\ell}$$

This is complete nonsense, for it insists that every ratio of real numbers $\underline{m} : \underline{n} = a : b$ is *rational*!

The Pythagorean commensurability supposition stems from their basic tenets: all is number (including length ratios), and the design of the gods is perfect (number means positive integer). The discovery of incommensurable ratios produced something of a crisis; a possibly apocryphal story states that a disciple named Hippasus (c. 500 BC) was set adrift at sea as punishment for its revelation.

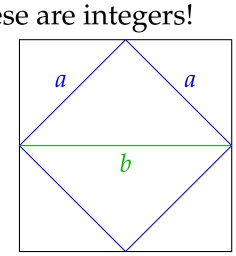
By 340 BC, however, the Greeks were happy to state that incommensurable lengths exist.

Theorem (Aristotle). *If the diagonal and side of a square are commensurable, then odd numbers equal even numbers.*

Inferred proof. In Socrates' doubled-square, suppose that side : diagonal $= a : b$; these are integers!

Assume at least one of a or b is odd, else the common sub-length may be doubled. The larger square is twice the smaller, whence the square numbers have ratio

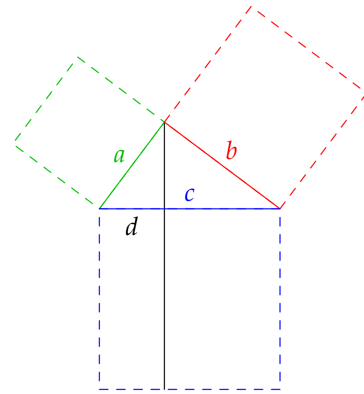
$$b^2 : a^2 = 2 : 1$$



It follows that b^2 is even and thus divisible by 4. But then a^2 is also even, whence *both* a, b are even. Whichever of a, b was odd is also even: contradiction! ■

Note the similarity of this argument to the modern proof of the irrationality of $\sqrt{2}$.

While there is no evidence that the Pythagoreans ever provided a correct proof of their famous Theorem, one argument possibly attributable to them relied on the flawed notion of commensurability.



'Proof' of the Pythagorean Theorem. Label the right-triangle a, b, c and drop the altitude to the hypotenuse c as shown. Let d be the part of the hypotenuse under a. Similar triangles tell us that

$$a : d = c : a \implies a^2 : ad = cd : ad \implies a^2 = cd$$

Thus the square on a has the same area as the rectangle below the d-side of the hypotenuse. Repeat the calculation on the other side to obtain $b^2 = c(c-d)$, and sum to complete the proof. ■

Since the only *numbers* were integers, a, b, c, d denote *integer multiples* of an assumed common sub-length. This restriction destroys the generality of the argument, since most triangles cannot be so labelled.
Book I of the *Elements* is plainly organized so as to provide a rigorous repair of the above argument which did not depend on the flawed notion of commensurability. With our modern understanding of real numbers and continuity, there is nothing wrong with the above.

Theaetetus of Athens (417–369 BC) Theaetetus is likely the source for much of the most difficult parts (Books VII & X) of the *Elements*. His essential definition of (in)commensurability comes from applying what is now known as the Euclidean algorithm to line segments.

Definition. Let $\underline{a} > \underline{b}$ be lengths/segments.[11] Repeatedly apply the division algorithm:

$$\begin{aligned} \underline{a} &= q_1\underline{b} + \underline{r}_1 & \underline{r}_1 &< \underline{b} & &(\exists q_1 \in \mathbb{N}_0 \text{ and a length } \underline{r}_1 < \underline{b}) \\ \underline{b} &= q_2\underline{r}_1 + \underline{r}_2 & \underline{r}_2 &< \underline{r}_1 \\ \underline{r}_1 &= q_3\underline{r}_2 + \underline{r}_3 & \underline{r}_3 &< \underline{r}_2, \quad \text{etc.} \end{aligned}$$

We say that $\underline{a}$ and $\underline{b}$ are *commensurable* if the algorithm terminates: some remainder $\underline{r}_n$ divides exactly into $\underline{r}_{n-1}$. Otherwise $\underline{a}$ and $\underline{b}$ are *incommensurable.*
Ratios are *equal* $\underline{a} : \underline{b} = \underline{c} : \underline{d}$ precisely when the sequences of quotients in the algorithm are equal.

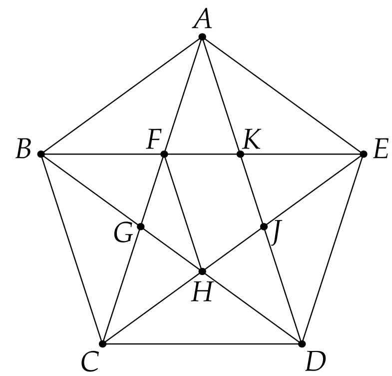
If $\underline{a}$ and $\underline{b}$ are commensurable, then $\underline{r}_n$ is their *greatest common sub-length*. If we write $\underline{a} = a\underline{r}_n$ and $\underline{b} = b\underline{r}_n$ for some integers a, b and rewrite the algorithm in the modern fashion, the result is the standard Euclidean algorithm computation of $\gcd(a, b) = 1$.

Example 1 $37 : 13 = 148 : 52$ since we obtain the same sequence of quotients $(2, 1, 5, 2)$:

$\underline{37} = 2 \cdot \underline{13} + \underline{11}$	$\underline{148} = 2 \cdot \underline{52} + \underline{44}$
$\underline{13} = 1 \cdot \underline{11} + \underline{2}$	$\underline{52} = 1 \cdot \underline{44} + \underline{8}$
$\underline{11} = 5 \cdot \underline{2} + \underline{1}$	$\underline{44} = 5 \cdot \underline{8} + \underline{4}$
$\underline{2} = 2 \cdot \underline{1}$	$\underline{8} = 2 \cdot \underline{4}$

[11]Thus '$\underline{a}$ is longer than $\underline{b}$' is a statement about lengths, not numbers. *Only* the quotients q_k need be integers.

Example 2 We sketch a proof that the side $\overline{AB}$ and diagonal $\overline{AC}$ of a regular pentagon $ABCDE$ are incommensurable.



1. Prove that $\triangle BAG$ is isosceles (just count angles!).
2. Take $a = |AC|$ and $b = |AB| = |AG|$. The first line of the algorithm reads

   $$|AC| = |AG| + |GC| = |AB| + |GC|$$

   so we write $\underline{a} = q_1\underline{b} + \underline{r}_1$ where $r_1 = |GC|$ and $q_1 = 1$.
3. Since $|GC| = |AF|$, the second line of the algorithm reads

   $$|AG| = |AF| + |FG| = |GC| + |FG|$$

   so that we again have a quotient of $q_2 = 1$.
4. Appealing to congruent isosceles triangles $\triangle DCG \cong \triangle EHF$ we see that $|GC| = |FH|$ is the diagonal of the interior regular pentagon. The third line of the algorithm is therefore the same as the first: we are back to considering the ratio of the diagonal to the side of a regular pentagon. The algorithm therefore continues forever with all quotients being 1.

Example 3 In modern language, the diagonal to the side of a square is the incommensurable ratio $\sqrt{2} : 1$. We apply Theaetetus' algorithm:

$$\underline{\sqrt{2}} = 1 \cdot \underline{1} + (\underline{\sqrt{2}-1})$$
$$\underline{1} = 2 \cdot (\underline{\sqrt{2}-1}) + (\underline{3-2\sqrt{2}})$$

Observe that $3 - 2\sqrt{2} = (\sqrt{2}-1)^2$, whence the second line reads $1 = 2x + x^2$. The following lines in the algorithm may therefore be obtained by repeatedly multiplying by x:

$$x = 2x^2 + x^3, \quad x^2 = 2x^3 + x^4, \quad \text{etc.},$$

resulting in a never-ending sequence[12] of quotients: $1, 2, 2, 2, 2, 2, 2, \ldots$

Eudoxus of Knidos (c. 390–337 BC) Eudoxus was arguably the most prolific pre-Euclidean mathematician. Apart from attending and perhaps teaching at Plato's academy, he is famous for explaining how to calculate with ratios of lengths (segments). For example:

Definition. $A : B > C : D$ if there exist positive integers m, n such that $mA > nB$ and $mC \le nD$.

At first glance it appears as if Eudoxus is telling us how to compare *rational* numbers; if A, B, C, D are integers, we see that

$$\frac{A}{B} > \frac{n}{m} \ge \frac{C}{D}$$

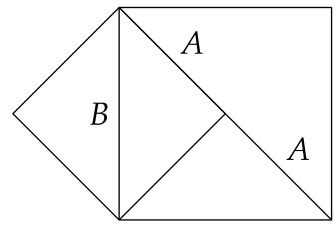
which is trivially satisfied by taking $m = D$ and $n = C$. To Eudoxus however, A, B, C, D could also be interpreted as *segments*. Building on the work of Theaetetus, his mathematics told the Greeks how to approximate incommensurable ratios with rational ratios.

[12] If you're interested in number theory, investigate the relationship of Theaetetus' algorithm to continued fractions…

Examples 1. To see that $13 : 3 > 17 : 4$, simply choose $m = 4$ and $n = 17$ to obtain

$$4 \cdot 13 = 52 > 51 = 3 \cdot 17$$

2. We show that the side : diagonal of a square is greater than 1 : 2; equivalently, the diagonal is less than twice the side.
   In the picture, side : diagonal $= A : B$. Choosing $m = D = 2$ and $n = C = 1$, we see that



$$mA = 2A > B = nB \qquad \text{(diag > side of large square)}$$

In modern language, this is merely $\frac{1}{\sqrt{2}} > \frac{1}{2}$.

Zeno of Elea (c. 450 BC) Zeno's arguments have provided philosophical fodder for millennia. Several illustrate the essential difficulties of the infinite and the infinitesimal that lie at the heart of the controversy around the development of calculus. Here are two of the most famous:

Achilles and Tortoise Achilles chases a Tortoise. After time t_0, Achilles reaches the Tortoise's starting position, but the Tortoise has moved on. After another time t_1, Achilles reaches the Tortoise's second position; again the Tortoise has moved. In this manner Achilles spends $t_0 + t_1 + t_2 + \cdots$ in the chase. Zeno's paradoxical conclusion is that Achilles never catches the Tortoise.

This paradox may be resolved (see Exercise 8), at least assuming both Achilles and the Tortoise travel at constant speeds: even though it be split into infinitely many subintervals of time, the total duration of the chase can expressed as the (finite) value of a convergent *infinite series.*

Arrow paradox An arrow is shot from a bow. At any given instant the arrow doesn't move. If time is made up of instants, then the arrow never moves.

This time Zeno debates the the idea that a finite time period can be considered as a sum of infinitesimal instants. The same difficulty is central to *integration.*

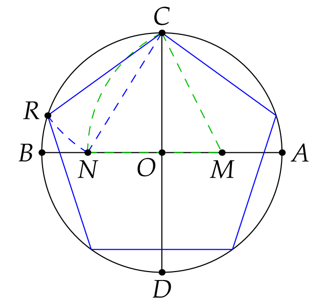
Constructions and Geometry By the mid 5^{th} century BC, Greek mathematicians were solving geometric problems using *ruler-and-compass* (peg-and-cord) constructions. This approach could have come to Greece from India, or might have arisen organically. Constructions were based on three rules, which became the first three postulates (axioms) of Book I of Euclid's *Elements.*

1. Two points may be joined by a straight line segment.
2. Any segment may be extended indefinitely.
3. Given a center and radius, one may draw a circle.

Theorems were often stated as *problems*: e.g., *to bisect a given angle.* A proof provided first a *construction,* then an argument justifying that the construction really had solved the problem.

By the time of Euclid, the Greeks knew how to construct an equilateral triangle, a square and a regular pentagon in a given circle.

Example We construct a regular pentagon in a given circle.[13]



1. Draw perpendicular diameters $\overline{AB}$ and $\overline{CD}$ and bisect $\overline{OA}$ at M.
2. Draw an arc centered at M with radius $|CM|$. Let N be the intersection of this arc with $\overline{OB}$.
3. Draw an arc centered at C with radius $|CN|$. Let R be the intersection of this arc with the original circle.
4. Move $|CR|$ around the circle to create a regular pentagon.

A purely geometric proof of the validity of this construction is too difficult for us, but you can easily check it using a calculator, Pythagoras' and the cosine rule: if the circle has radius 2, then

$$|CR|^2 = |CN|^2 = |ON|^2 + |OC|^2 = (\sqrt{5}-1)^2 + 2^2 = 10 - 2\sqrt{5} = 2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cos 72°$$

Construction problems have motivated mathematicians ever since. In 1796, Gauss (then 19) constructed a regular 17-gon. A classification of constructable regular polygons took until 1837.

Theorem. *A regular n-gon is constructable if and only if $n = 2^k F_1 \cdots F_r$ where $F_1, \ldots, F_r$ are distinct primes of the form $2^{(2^n)} + 1$.*

After the 17-gon, the next prime-sided constructable n-gon has $257 = 2^{2^3} + 1$ sides!

By 400 BC, the Greeks were referencing the second and third *impossible constructions of antiquity*:

1. Trisecting a general angle.
2. Doubling (the volume of) a given cube.
3. Squaring a circle (construct a square with the same area as a given circle).[14]

It wasn't until the advent of field theory in the 1800s that such were proved to be impossible using ruler-and-compass constructions.

Summary Several of the mathematical techniques in this section are difficult, and the results technical. It isn't important to become proficient with all of these ideas! Instead play with them to help develop an appreciation of two overarching points:

1. Even before Euclid, the focus of Greek mathematics was more abstract and less practical than other ancient cultures (Egyptians, Babylonians, Chinese, etc.), largely due to the influence of wider Greek philosophy and religion. The modern liberal arts ideal of learning for its own sake—to celebrate the beauty of knowledge and to expand the mind—is, to a large extent, a Greek inheritance.

2. The ancient Greeks pondered fundamental mathematical questions and concepts—*number* versus *length*, continuity, irrationality, infinitesimals, constructions—ideas that have stimulated mathematical research ever since. These particular issues would not rigorously be resolved until the 1800s when luminaries such as Gauss, Cauchy and Riemann developed modern analysis and algebra.

[13]Theorem IV. 11 of the *Elements* presents a less practical construction. Ours follows from Theorem XIII. 10: if a regular pentagon, hexagon and decagon are inscribed in a circle, then their sides form a right-triangle.

[14]"You can't square that circle" is now a metaphor for something that can't be done.

Exercises 3.2. 1. Construct five Pythagorean triples using the formula $\left(n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right)$ where n is odd. Construct five more using the formula $\left(m, \left(\frac{m}{2}\right)^2 - 1, \left(\frac{m}{2}\right)^2 + 1\right)$ where m is even.

2. Suppose $2^n - 1 = p$ is prime (its only positive divisors are itself and 1). List the positive divisors of $2^{n-1}(2^n - 1)$ and hence prove Theorem IX. 36.

3. Draw a *picture* with dots to show that eight times any triangular number plus 1 makes a square, and that any odd square diminished by 1 becomes eight times a triangular number. That is:

   (a) $8 \cdot \frac{1}{2}m(m+1) + 1$ is a perfect square.

   (b) If n is odd, then $n^2 - 1 = 8 \cdot \frac{1}{2}m(m+1)$ for some m.

4. Find a construction (using the *ruler-and-compass* constructions) to bisect a given angle, and show that it is correct.

5. Sketch a construction inscribing a regular hexagon in a circle.

   (*Assume you can construct an equilateral triangle on a given segment—Thm I. 1 of Euclid, pg. 27*)

6. (A line-doubling paradox) One line has twice the length of another and so has *more* points. However, there is a bijective correspondence between the points on these lines; the two lines therefore have the *same number* of points.

   Explain the second observation. How can you resolve the paradox?

7. The *cycle of fifths* is a musical concept stating that twelve perfect fifths equals seven octaves (pg. 18). State this claim *numerically*, and show that it is a contradiction.

   (*Hint: two strings are seven octaves apart if their lengths are in the ratio* $2^7 : 1$)

8. We use modern language to resolve Zeno's paradox of Achilles and the Tortoise. Suppose Achilles travels at speed v_A, the tortoise at speed $v_B < v_A$, and that the tortoise starts a distance d ahead of Achilles.

   (a) Prove that $t_n = \frac{d}{v_A}\left(\frac{v_T}{v_A}\right)^n$ for each positive integer n.

   (b) Compute $\sum_{n=0}^{\infty} t_n$ using the geometric series formula from calculus.

   (c) Verify the time-value computed in (b) as would a modern Physicist; by considering the motion of Achilles relative to the tortoise.

9. Use Theaetetus' definition of equal ratios to prove that $46 : 6 = 23 : 3$.

10. (Hard) A line of length 1 is divided at x so that $\frac{1}{x} = \frac{x}{1-x}$. Prove that 1 and x are incommensurable. Indeed, show that $1 : x$ is *the same* as diagonal : side of a regular pentagon.

    (*Hint: the first line of the algorithm is* $1 = x + x^2 \ldots$)

11. (Hard) Let $a > b$ and c be *positive lengths.* Use Eudoxus' definition to *prove* that $c : b > c : a$.

    (*Hint: let* n *be the smallest integer such that* $n(a-b) \geq c$*; its existence is the "archimedean property"*)

3.3 Euclid and the *Elements*

Euclid worked in the Library of Alexandria, named for the Greek general Alexander the Great who conquered Egypt in 323 BC. The Library was constructed around 320 BC as a means of organizing the knowledge of the world and for the demonstration of Greek power. Although it was seriously damaged on several occasions, the Library remained a center of scholarship until around AD 500. Below is a map of the city around AD 400: note the size and centrality of the Library.

It is hard to argue against Euclid's *Elements* (c. 300 BC) as the most influential mathematics text ever produced. Likely a compilation of earlier mathematical work rather than a pure original, it was edited and added to over the centuries, eclipsing and subsuming other works. Particular import were the edits of Theon of Alexandria (c. AD 400) and his daughter Hypatia, both prolific scholars in their own right. Due to edits such as these, the precise contents of the original are unknown.

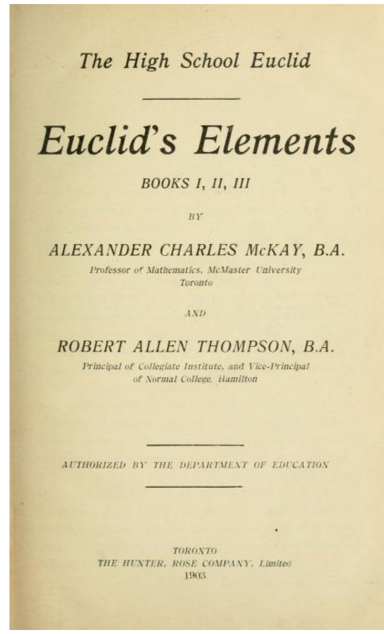
Extant fragments date to around AD 100. The earliest (almost) complete copy is from the the 9^{th} century; written in Greek and held at the Vatican, it is missing some of the edits of Theon & Hypatia, thus demonstrating that multiple versions were in circulation.

Earliest Fragment c. AD 100

Full copy 9^{th} C

Until the mid 20th century, some version of the *Elements* would have been used as a high-school textbook in most western and middle-eastern countries. Many editions and variations have been produced, four of which are shown below:

Latin translation, 1572


The High School Euclid

Euclid's Elements

BOOKS I, II, III

BY

ALEXANDER CHARLES McKAY, B.A.

Professor of Mathematics, McMaster University
Toronto

AND

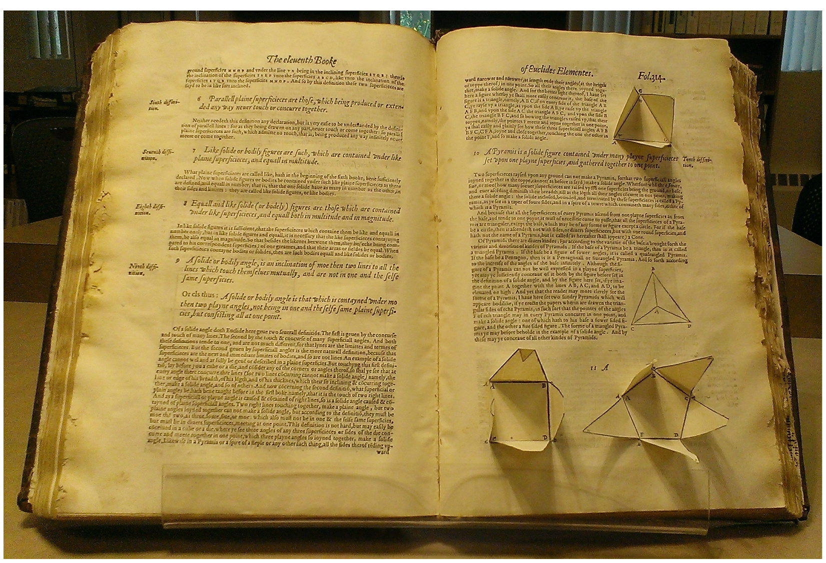
ROBERT ALLEN THOMPSON, B.A.

Principal of Collegiate Institute, and Vice-Principal
of Normal College, Hamilton

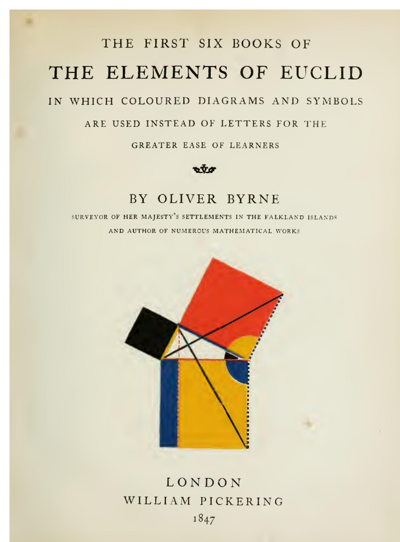
AUTHORIZED BY THE DEPARTMENT OF EDUCATION

TORONTO
THE HUNTER, ROSE COMPANY, Limited
1903

High School textbook, 1903



Pop-up edition, 1500s


THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

IN WHICH COLOURED DIAGRAMS AND SYMBOLS
ARE USED INSTEAD OF LETTERS FOR THE
GREATER EASE OF LEARNERS

BY OLIVER BYRNE

SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS
AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS

LONDON
WILLIAM PICKERING
1847

Color edition, 1847

You can download Byrne's color edition here (very large file!). It is notably different from many earlier editions: it contains a much longer list of definitions, inserts many more axioms, and relabels propositions 4 and 5 as axioms (pages xviii–xxiii). The picture on the cover page is Euclid's proof of Pythagoras' Theorem (Book I, Thm. 47).

A Brief Overview of the *Elements*

The *Elements* consists of thirteen books covering two- and three-dimensional geometry, computations and number theory. Whether some version of every book was written by Euclid himself is unknown.[15] The key feature of the *Elements* is its *axiomatic* presentation. Each book begins with a list of axioms/postulates and definitions and proceeds to prove theorems deduced from these. This *axiomatic method* is essentially universal in modern mathematics, and its advent is fundamentally what sets Greek mathematics apart from everything that came before.

We briefly discuss Book I, then give some flavor of the remainder of the text with a few example results. Several examples of material from later books were mentioned in the previous section.

Book I Consists of 48 theorems, culminating with Pythagoras' and its converse. It seems likely that Euclid organized Book I with the goal of proving this important result in a rigorous manner: recall (pg. 20) how the Pythagorean 'proof' relied on the erroneous notion of commensurability. Here are the postulates from Book I: the first three are what define ruler-and-compass constructions (pg. 22).

- P1 Given any two points, a straight line can be drawn between them
- P2 Any line may be indefinitely extended
- P3 Given a center and a radius, a circle may be drawn
- P4 All right angles are equal to each other
- P5 If a straight line crosses two others so that the angles on the same side make less than two right angles, then the two lines meet on that side of the original.

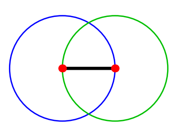
The fifth postulate is awkwardly phrased. An equivalent modern statement is *Playfair's axiom*:

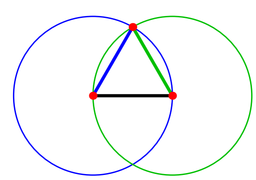
> There is at most one parallel through a given point not on line.

For centuries, mathematicians tried to prove that this postulate was a theorem of the others until; it was eventually shown to be necessary with the advent of hyperbolic geometry in the 1800s. Euclid's refusal to use the parallel postulate until Theorem 29 suggests he understood this awkwardness.

Theorems are generally presented as *problems*: a pictorial construction is provided, then Euclid proves that the construction really solves the problem. Here is Euclid's first theorem.

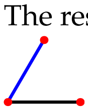
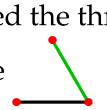
Theorem (I. 1). *Problem: To construct an equilateral triangle on a given segment.*

Proof. Given construct two circles (P3) 



Join one of the circle intersections to the endpoints of the original segment (P1)

The result is an equilateral triangle; indeed the three sides are congruent, for

 are radii of a common circle, as are  ∎

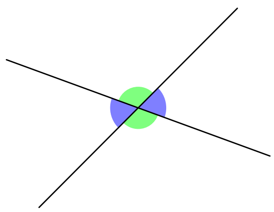
[15] It is not even certain that Euclid was a single person as opposed to a figurehead or a name for a collective. It is hardly surprising that we know so little about someone who lived 2300 years ago. The same questions are sometimes raised about William Shakespeare who lived only 400 years ago!

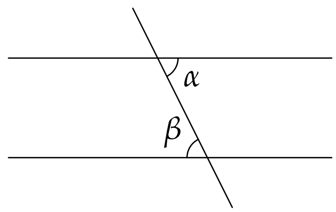
After this Euclid proceeds to establish several well-known results. Since this isn't a geometry class, we'll omit most of the details. You can find more of these here, in Byrne, or elsewhere.

Thm I. 4 Side angle side: if triangles have two pairs of congruent sides and the angles between them are also congruent, then the remaining sides and angles are congruent.

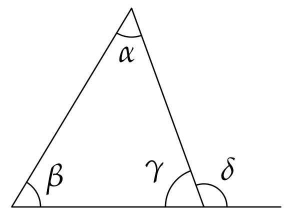
Thm I. 15 Vertical angles: if two lines meet, then the opposite angles made are congruent.

Thms I. 27 & 29 Angles and parallels: if a line falls on two other lines, then the two lines are parallel if and only if the alternate angles are congruent ($\alpha \cong \beta$ in the picture).

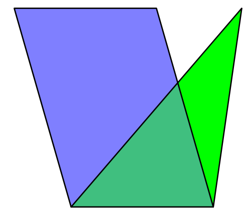

Thm I. 15



Thms I. 27/29



Thm I. 32


Thm I. 41

Thm I. 32 Angle sums in a triangle: if one side of a triangle is protruded, the exterior angle equals the sum of the opposite interior angles.

In the picture $\alpha + \beta = \delta$; in modern language $\alpha + \beta + \gamma = 180°$.

Thm I. 41 A parallelogram and triangle on the same base and with the same height have area in the ratio 2:1.

The last two results of Book I are Pythagoras and its converse.

Theorem (I. 47). *The square on the hypotenuse of a right-triangle has area equal to the sum of the areas of the squares on the remaining sides.*

Proof. Given a right-angle at A, drop the perpendicular from A across $|BC|$ to L.

$\triangle FBC$ and $\square ABFG$ share the same base $\overline{BF}$ and height $\overline{AB}$. By Thm I. 41,

$$\text{area}(\square ABFG) = 2\,\text{area}(\triangle BCF)$$

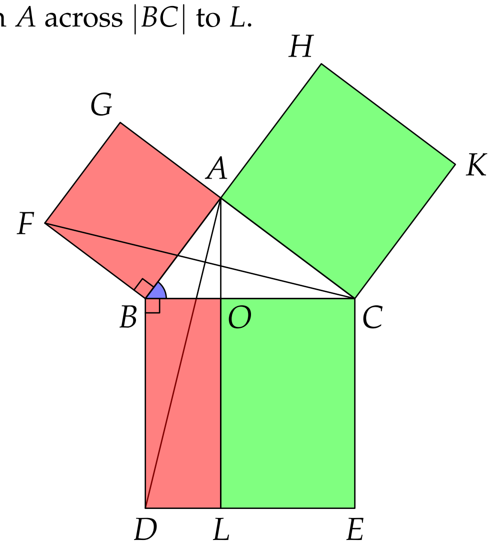
Similarly (base $\overline{BD}$, height $\overline{BO}$)

$$\text{area}(\square BOLD) = 2\,\text{area}(\triangle ABD)$$

Side-Angle-Side (Thm I. 4) $\implies$ $\triangle ABD \cong \triangle FBC$; the triangles have the same area, and so

$$\text{area}(\square ABFG) = \text{area}(\square BOLD)$$

Similarly $\text{area}(\square ACKH) = \text{area}(\square OCEL)$.



∎

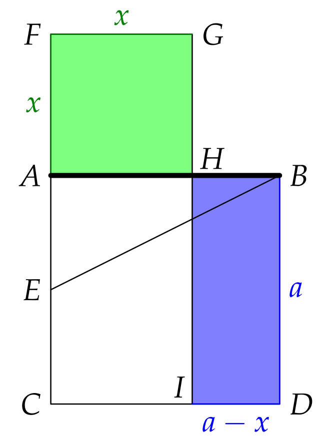
The converse (Thm I. 48) is Exercise 3.

Book II Geometric solutions to problems that would now be treated using algebra. Much of this is attributable to the Pythagoreans.

(Thm II. 11) A segment may be divided so that the rectangle contained by the whole and one of the sub-segments is equal to the square on the remaining sub-segment.

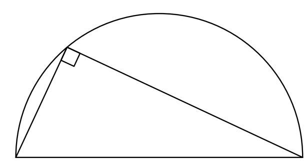
We rephrase this in modern language. Suppose the given segment $\overline{AB}$ has length a, our goal is to find H on $\overline{AB}$ such that $|AH| = x$ and $x^2 = a(a-x)$. Euclid is providing a *geometric solution* to a quadratic equation!

1. Construct $\square ABDC$ on $\overline{AB}$
2. Let E be the midpoint of $\overline{AC}$ and connect $\overline{EB}$
3. Extend $\overline{AC}$ and lay off $\overline{EF} \cong \overline{EB}$ on $\overline{AC}$ extended
4. Construct $\square AFGH$ and drop the perpendicular to I
5. To finish: prove that area$(\square AFGH)$ = area$(\square BDIH)$



The solution $x = |AH| = \frac{\sqrt{5}-1}{2}a$ means that $\overline{AH} : \overline{HB}$ is the *golden ratio*.

Book III Theorems regarding circles and tangency. Most of this material likely came from Hippocrates.
(Thm III. 31) Thales' Thm: a triangle in a semi-circle is right-angled.



Book IV Construction of regular 3, 4, 5, 6 and 15-sided polygons inscribing and exscribing a circle. Also likely the work of Hippocrates.

Book V Ratios and magnitudes à la Eudoxus.

(Thm V. 11) If $a : b = c : d$ and $c : d = e : f$ then $a : b = e : f$

Book VI Ratios of magnitudes applied to geometry (similarity results).

(Thm VI. 4) Triangles with equal angles have corresponding sides proportional.

(Thm VI. 8) The altitude from the right angle of a right triangle divides it into two triangles similar to each other and the the original.

(Thm VI. 31) Corrected Pythagorean proof (pg. 20) of Pythagoras' using the proportions of Eudoxus and Thm VI. 8.

Book VII Divisibility and the Euclidean algorithm. Probably due to the Pythagoreans and Theaetetus.

Book VIII Number progressions, geometric sequences. Possibly due to studies in music by Archytas (a Pythagorean who taught Plato mathematics).

Book IX Number Theory: even/odd + perfect numbers.

(Thm IX. 20) There are infinitely many primes.

Book X Discussion of commensurable and incommensurable ratios. Long and difficult, possibly derived from Theaetetus.

Book XI Solid geometry (lines/planes in 3D).

(Thm XI. 28) A parallelepiped is bisected by its diagonal plane.

Book XII Ratios of areas and volumes (Eudoxus).

(Thm XII. 2) The areas of circles are in the same ratio as the squares on their diameters.

Book XIII Construction of regular polyhedra inside a sphere and their classification.

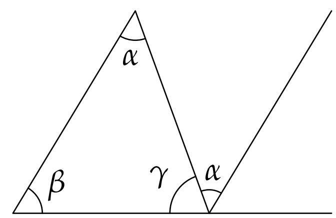
(Thm XIII. 10) If a regular pentagon, hexagon and decagon are inscribed in the same circle, then their sides form a right-triangle.

One could study the *Elements* and its influence for a lifetime and not be done! Hopefully this very brief overview convinces you why the book had such a profound impact on mathematics.

Exercises 3.3. 1. Prove Thales' Theorem (III. 31) (pg. 29).

(*Hint: start by joining the center of the circle to the apex of the triangle...*)

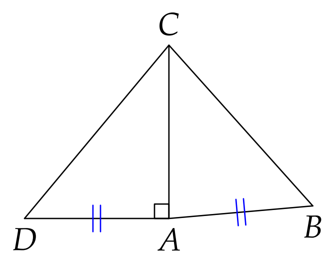
2. Use the picture to provide a proof of Thm I. 32: the sum of the three interior angles of a triangle is equal to two right angles.
Show that the proof depends on Thm I. 29, and therefore on the parallel postulate.
(*This isn't quite the same as Euclid's argument*)



3. Suppose that the square on side $\overline{BC}$ of $\triangle ABC$ has the same area as the sum of the squares on the other sides $\overline{AB}$, $\overline{AC}$. As in the picture, draw a perpendicular $\overline{AD} \cong \overline{AB}$.

   (a) Explain why $\overline{DC} \cong \overline{BC}$.

   (b) Hence conclude that $\triangle ABC$ is right-angled at A.



4. Prove Thm III. 3: A diameter of a circle bisects a chord if and only if it is perpendicular to the chord.

5. Verify that Euclid's construction for Thm II. 11 really does solve the given problem.

   (*You can use modern algebra!*)

6. Draw a semi-circle with diameter $9 + 5 = 14$. Solve the equation $\frac{9}{x} = \frac{x}{5}$ *geometrically*, by constructing a vertical line whose length is x.

7. Show that areas of similar segments of circles are proportional to the squares of the length of their chords.

   (*You may assume that areas of circles are proportional to the squares on their diameters and can use modern algebra/trigonometry if you wish*)

3.4 Archimedes of Syracuse

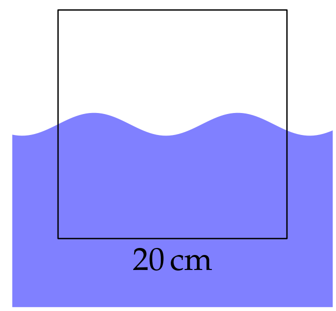
Archimedes (287–212 BC) is arguably the greatest ancient mathematician. Syracuse is on the island of Sicily at the foot of the Italian peninsula; at the time of Archimedes' birth this was a Greek city-state, though under threat from the expanding Roman Empire. Archimedes famously helped defend Syracuse against the Romans using catapults, though he ultimately died at their hands after the city fell. He is believed to have travelled to Alexandria in his youth and perhaps studied with scholars at the library, including Eratosthenes (pg. 34).

Archimedes' genius was practical not just mathematical. Beyond his anti-Roman catapults, he is credited with a large number of inventions and technical innovations, including *Archimedes' screw*, still used in modern irrigation systems to elevate water. He is acknowledged as the founder of *hydrostatics*, where *Archimedes' principle* states that an object immersed in water loses weight equal to that of the displaced water. A famous story recounts Archimedes using this to detect whether a smith had used all the gold he had been given in the manufacture of a crown.

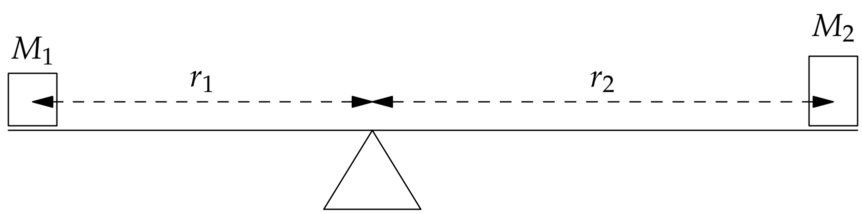
Example A cube with side length 20 cm floats such that the waterline is halfway up the cube. By Archimedes' principle, the weight of the cube is the same as that of the volume of displaced water:

$$20 \times 20 \times 10 = 4000\,\text{cm}^3$$

which has a weight (mass) of roughly 4 kg.



Levers Archimedes made great study of levers, both for practical purposes and as a method of calculation.



Given masses M_1, M_2 located distances r_1, r_2 from a pivot, Archimedes states:

- The lever balances $\iff M_1 : M_2 = r_2 : r_1$
- The lever rotates clockwise $\iff M_1 : M_2 < r_2 : r_1$
- The lever rotates counter-clockwise $\iff M_1 : M_2 > r_2 : r_1$

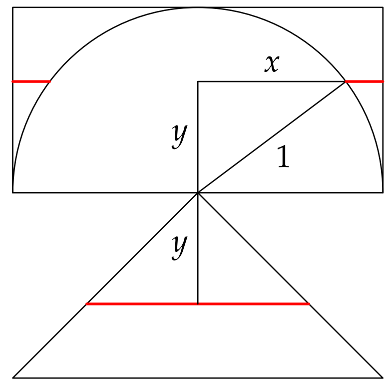
In modern terms we'd compare the *torques* $\tau_1 = M_1 r_1$ and $\tau_2 = M_2 r_2$. Since torque requires the multiplication of non-numerical quantities, Archimedes would instead have considered this using Eudoxus' theory of proportions.
For example, to find the mass M_2 required to balance a lever given $M_1 = 12\,\text{lb}$, $r_1 = 4\,\text{ft}$ and $r_2 = 3\,\text{ft}$, Archimedes would have observed that

$$M_2 : M_1 = 4 : 3 \implies M_2 = 16\,\text{lb}$$

***The Method*: is Archimedes the founder of calculus?** A previously unknown work of Archimedes was discovered in 1899. As an amazing application of the lever principle, Archimedes makes an argument that looks remarkably like modern calculus; he could be claimed to be its earliest practitioner by 1800 years! The method was outlined in a letter to Eratosthenes and includes part of an argument for proving Archimedes' favorite theorem, a picture of this result was engraved on his tomb.

Theorem. *A cone, hemisphere and cylinder with the same base and height have volumes in the ratio* 1 : 2 : 3*. Using modern formulæ, if the height is r, then the volumes are $\frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3$.*

Here is a modernized version illustrating Archimedes' approach. Suppose the 'base' is a disk with radius 1, remove the hemisphere from the cylinder and place the cone beneath. Compare the cross-sections the same distance y from the apex of the cone.



- The circular cross-section of the cone has radius y whence its area is proportional to the square on the radius: πy^2.
- The upper annular cross-section has area proportional to the difference of the squares on the radius of the cylinder and on the distance x. By Pythagoras' the cross-sectional area is

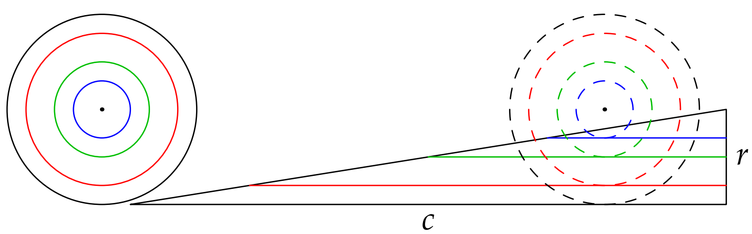
$$\pi(1^2 - x^2) = \pi(1 - (1 - y^2)) = \pi y^2$$

The cross-sections are therefore in balance with respect to a vertical lever whose pivot lies at the center of the picture. Archimedes concludes that the cone and the upper-figure are in balance: that is

$$V_{\text{cone}} = V_{\text{cylinder}} - V_{\text{hemisphere}}$$

Combining this with the fact that $V_{\text{cylinder}} = 3V_{\text{cone}}$ (e.g., Exercise 7d) gives the desired result.

Here is another argument of Archimedes' with a suggestion of calculus. A disk comprises infinitely many concentric circles, the circumference of each being proportional to its radius. 'Unwind' these circles to obtain a triangle; one side is the radius of the disk, the other its circumference. The area of a circle is therefore that of a triangle with sides the radius and circumference of the circle: $A = \frac{1}{2}rc$.



The Method includes several of these calculus-like discussions. While efficient, Archimedes felt that his approach didn't constitute a proof and provided alternative arguments elsewhere in his writings. The essential problem is this;

> Can we really say that an area *equals* its cross-sectional lines? Or that a volume *equals* its cross-sectional areas? Lines have no width so if we add them up we have no area. If they have width, then infinitely many of them have infinite area.

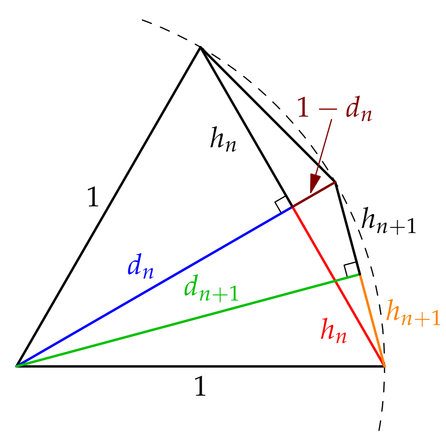
These are really variations of Zeno's paradoxes (pg. 22) regarding infinitesimals and indivisibles!

Archimedes' arguments would be resurrected in the early 1600s by Cavalieri and Galileo as the development of calculus gathered pace. The same duality of presentation characterised this later development: Newton and others found the infinitesimal approach efficient, but felt the need to present *geometric* proofs to convince readers that their results weren't mere trickery.

It is tempting to imagine what might have happened if Archimedes' *method* had been accepted and preserved as part of the Greek canon; if calculus had been developed 1800 years earlier, how might this have affected technological development? Would the space-race have happened in AD 500?!

Quadratures Archimedes also approximated areas and arc-lengths of various figures using limit-like argumentation. Here is how he approached the area/circumference of a circle.

1. Inscribe a regular hexagon in a circle (of radius 1 say) and compute its perimeter (6).
2. Halve each angle to obtain a regular dodecagon: compute its perimeter ($12\sqrt{2-\sqrt{3}}$).
3. Repeat the angle-halving process: Archimedes did this with 24-, 48- and 96-gons to obtain an increasing sequence of perimeters bounded above by the circumference of the circle (2π).
4. Repeat the same calculation with circumscribed polygons to obtain a decreasing sequence of over-estimates.
5. Using 96-sided polygons allowed Archimedes to obtain the estimate $3\frac{10}{71} < \pi < 3\frac{1}{7}$.



Archimedes' halving process relied on an induction step, an approximation of which we mimic here. Suppose we have an isosceles triangle with equal legs 1, altitude d_n, and chord $2h_n$. We halve the angle to find the new altitude d_{n+1} and chord $2h_{n+1}$. Everything follows from three applications of Pythagoras':

$$1 = d_n^2 + h_n^2$$
$$(2h_{n+1})^2 = h_n^2 + (1-d_n)^2$$
$$1 = d_{n+1}^2 + h_{n+1}^2$$

Expanding and cancelling, we obtain

$$d_{n+1}^2 = \frac{1}{2}(1+d_n), \qquad h_{n+1}^2 = 1 - d_{n+1}^2 = \frac{1}{2}(1-d_n)$$

Since $d_0 = \frac{\sqrt{3}}{2}$ and $h_0 = \frac{1}{2}$, we may compute the entirety of both sequences:

$$d_1 = \frac{1}{2}\sqrt{2+\sqrt{3}}, \quad d_2 = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}, \quad \ldots \quad d_n = \frac{1}{2}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{3}}}}$$
$$h_1 = \frac{1}{2}\sqrt{2-\sqrt{3}}, \quad h_2 = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}, \quad \ldots \quad h_n = \frac{1}{2}\sqrt{2-\sqrt{2+\cdots+\sqrt{2+\sqrt{3}}}}$$

where the n^{th} terms have n copies of the digit 2 under the square-root. The circumference and area of the $6\cdot 2^n$-sided polygon inscribed in the circle are therefore

$$C_n = 12\cdot 2^n h_n, \qquad A_n = 6\cdot 2^n d_n h_n = 6\cdot 2^{n-1} h_{n-1} = \frac{1}{2}C_{n-1}$$

These sequences increase to 2π and π respectively. For a 96-sided polygon, Archimedes would have had to approximate

$$C_4 = 12 \cdot 2^4 h_4 = 96\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} > 6.282 \implies \pi > 3.141 > 3\frac{10}{71}$$

Other Highlights of the later Greek Period: 300 BC–AD 500

We'll consider ancient astronomy, including Greek contributions, in the next chapter. Here are a few of the other developments of the late Greek period and some historical context.

- Eratosthenes (276–194 BC) grew up in Cyrene (c. 500 miles west of Alexandria in modern-day Libya) and moved to Alexandria in adulthood to become its librarian. He is credited with a simple algorithm for finding primes: the *Sieve of Eratosthenes.*
  - List the integers $n \geq 2$.
  - Leave 2 and delete all its multiples.
  - Leave 3 and delete its multiples.
  - Repeat ad infinitum: each time one reaches a number, leave it and delete its multiples.
  - The remaining list contains all the primes.
- Apollonius (225 BC) writes an eight-volume book on conic sections building on earlier work of Menaechmusus (350 BC).
- By 146 BC the Greek empire had fallen under Roman rule. Alexandria remained important. Educated Greeks still spoke and wrote in Greek rather than (Roman) Latin. For context, Julius Caesar ruled Rome around this time (died 44 BC).
- Heron (AD 75) proves the formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of triangle, where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. This was likely known to Archimedes; Heron's work was a compilation of earlier mathematics.
- Around AD 100 the Neopythagorean's worked in Alexandria, studying music, philosophy, and number, with the intent of reviving the teachings of Pythagoras.
- Around AD 400, Theon and Hypatia produce the most widely-read edition of Euclid's *Elements* as well as improving upon several earlier mathematical topics.
- In AD 395 the Roman empire split into eastern and western parts centered on Rome and Byzantium/Constantinople. The western empire rapidly declined under the pressures of corruption and barbarian attacks, collapsing completely by AD 500. Alexandria experienced riots and a bloody power-struggle (Hypatia was murdered by a mob in 415) and the library of Alexandria was severely damaged and possibly destroyed at this time. In 642, Alexandria was captured by the new Islamic caliphate. Much of the material in the library survived by being copied and transported to various places of learning; particularly Constantinople and Baghdad. For the next 600 years, the knowledge of Alexandria was largely a mystery to (western) Europe.

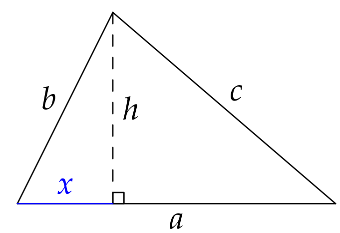
Exercises 3.4. 1. If a weight of 8 kg is placed 10 m from the pivot of a lever and a weight of 12 kg is placed 8 m from the pivot in the opposite direction, toward which weight will the lever incline? Answer using Archimedes' language.

2. Use Eratosthenes' Sieve to find all the primes < 100.

3. (a) Prove Heron's formula as follows.

   i. Let h be the altitude and x the base of the left-hand right-triangle. Apply Pythagoras' to the two right-triangles to show that

$$x = \frac{a^2 + b^2 - c^2}{2a}$$



   ii. Substitute in $h^2 = b^2 - x^2$ to find h in terms of a, b, c and thus deduce Heron's formula.

   (b) Find the area of a triangle with sides 4, 7 and 10.

4. Suppose C_n is the circumference of a $6 \cdot 2^n$-sided inscribed polygon in a unit circle. Show that the circumference of the corresponding *circumscribed* polygon is $C_n^{\text{ex}} = \frac{1}{d_n} C_n$.

5. Use the modern formula $A = \frac{1}{2}ab \sin C$ to prove that, for any $k \in \mathbb{N}$

$$\frac{1}{2} k \sin \frac{2\pi}{k} < \pi < k \tan \frac{\pi}{k}$$

   Moreover, explain why both sides converge to π.

6. Instead of modern algebra, Archimedes used several geometric lemmas to help find the areas of polygons inscribed in and circumscribing circles. Here is one; prove it!

   Let $\overline{OA}$ be the radius of a circle and $\overline{AC}$ be tangent to the circle at A. Let D lie on $\overline{AC}$ such that $\overline{OD}$ bisects $\angle COA$. Then

$$\frac{|DA|}{|OA|} = \frac{|CA|}{|CO| + |OA|} \quad \text{and} \quad |DO|^2 = |OA|^2 + |DA|^2$$

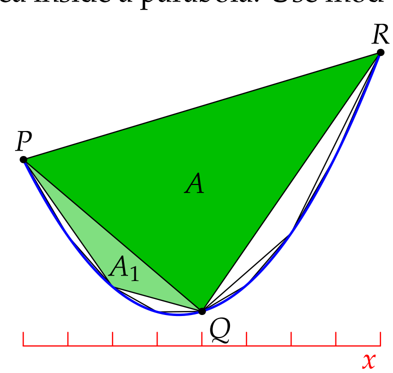
   (*Hint: draw a picture and let T be the intersection of the circle and $\overline{OC}$*)

7. Archimedes used a geometric series approach to evaluate the area inside a parabola. Use modern algebra for this question.

   (a) Suppose $y = a + bx + cx^2$ is the equation of a parabola. If P, Q, R have x co-ordinates in an arithmetic sequence $x - \epsilon, x, x + \epsilon$, show that the area of $\triangle PQR$ is $A = |c|\epsilon^3$; *independent of x*!

   (b) With reference to the picture, explain why the areas of the labelled triangles satisfy $A_1 = \frac{1}{8}A$.

   (c) Use a geometric series to prove that the area inside the parabola bounded by $\overline{PR}$ is $\frac{4}{3}A$



   (d) How might this result have been applied to show that the volume of a cone equals $\frac{1}{3}$ that of a cylinder with the same base and height?