Ancient Greek Mathematics

The Greek Empire

- Ancient Greek *civilization* generally accepted to date from around 800 BC. Primarily centered on the Aegean Sea (between modern-day Greece and Turkey) containing hundreds of islands and loosely affiliated city-states.
- Many wars between city-states other empires (e.g. Persians).
- By 500 BC covered much of modern Greece, the Aegean and southern Italy. As a trading/sea-faring culture, built/captured city-states (colonies/trading-outposts)all around the north and east coast of the Mediterranean from Spain round the Black Sea and Anatolia (modern Turkey) to Egypt.
- Alexander the Great (356–323 BC) extended empire around the eastern Mediterranean inland capturing mainland Egypt and then east to western India and Babylon where he died.
- Eventually becomes part of the Roman Empire c.146 BC though Romans left Greek largely essentially intact apart from crushing several rebellions.
- Greek civilization flourished even as the Rome collapsed, continuing as part of the Byzantine Empire.

Map of Greek Empire c.500 BC from timemaps.com

Ancient Greece is important for far more than just mathematics and one course cannot begin to do justice to it. Much of modern western thought and culture including philosophy, art logic and science has roots in Ancient Greece. While undeniably important, western culture has often over-emphasized the role of the Greeks and downplayed the contribution of other cultures to our inherited knowledge.

Mathematics and Philosophical Development

- Inquiry into natural phenomena encouraged through the personification of nature (sky = man, earth = woman) which pervaded early religion.
- By 600 BC 'philosophers' were attempting to describe such phenomena in terms of natural causes rather than being at the whim of the gods. For example, all matter was suggested to be comprised of the four elements (fire, earth, water, air).
- Development of Mathematics linked to religion (mysticism/patterns/assumption of perfection in the gods' design), philosophy (logic) and natural philosophy (description of the natural world). Mathematics elevated from purely practical considerations to an extension of logic: the Greeks were unhappy with approximations even when such would be perfectly suitable for practical use. Led to the development of axiomatics and proof.
- Limited extant mathematics from pre-300 BC. Most famous work is Euclid's *Elements* c.300 BC, indisputably the most important mathematical work in western mathematics and a primary textbook in western education until the early 1900's. Probably a compilation/editing of earlier works, its importance meant other works were sacrificed and sometimes subsumed by it.
- The word *theorem* (theory, theorize, etc.) comes from the Greek *theoreo* meaning *I contemplate.* A theorem is therefore an observation based on contemplation.
- Later mathematics included Ptolemy's *Almagest* c.150 AD on Astronomy and the forerunner of trigonometry: basis of western astronomical theory until the 1600's.

Enumeration

The ancient Greeks had two primary forms of enumeration, both developed c.800–500 BC.

Attic Greek (Attica = Athens): Strokes were used for 1–4. The first letter of the words for 5, 10, 100, 1000 and 10000 denoted the numerals. For example,

- $\pi\epsilon\nu\tau\epsilon$ (pente) is the Greek word for five, whence Π denoted 5.
- $\delta\epsilon\kappa\alpha$ (deca) means ten, so $\Delta = 10$.
- H (hekaton), X (khilias) and M (myrion/myriad) denoted 100, 1000 and 10000 respectively.
- Combinations were used, e.g. $\Delta\Delta\Pi\Pi||| = 223$.

The construction of large numbers was very similar to the more familiar Roman numeral system.

Ionic Greek (Ionia = middle of Anatolian coast): the alphabet denoted numbers 1–9, 10–90 and 100–900 in the same way as Egyptian hieratic numerals were formed. The alphabet differs from modern Greek due to three archaic symbols ϛ, ϙ, ϡ (stigma, qoppa, sampi).
Larger numbers used a left subscript to denote thousands and/or M (with superscripts) for 10000, as in Attic Greek. For example,

$$35298 = {}_{,}\lambda{}_{,}\varepsilon\sigma\text{ϙ}\eta = \overset{\gamma}{M}{}_{,}\varepsilon\sigma\text{ϙ}\eta$$

1	α	10	ι	100	ρ
2	β	20	κ	200	σ
3	γ	30	λ	300	τ
4	δ	40	μ	400	υ
5	ε	50	ν	500	ϕ
6	ϛ	60	ξ	600	χ
7	ζ	70	o	700	ψ
8	η	80	π	800	ω
9	θ	90	ϙ	900	ϡ

Eventually a bar was placed over numbers to distinguish them from words (e.g. $\overline{\xi\theta} = 89$). Modern practice is to place an extra superscript (*keraia*) at the end of a number: thus $35298 =$ ͵λ͵εσϙη′ Reciprocals/fractions were denoted with accents: e.g. $\overline{\acute{\theta}} = \frac{1}{9}$. The use of Egyptian fractions persisted in Europe into the middle ages.

Both systems were fine for record-keeping but terrible for calculations! Later Greek mathematicians, in particular Ptolemy, adapted the Babylonian sexagesimal system for calculation purposes thus cementing the use of degrees in astronomy and navigation.

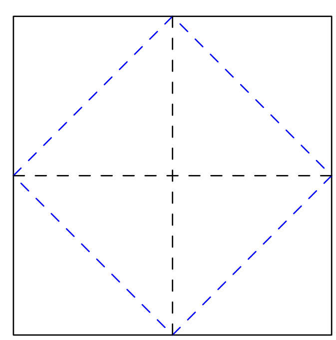
Early (pre-Euclidean) Greek Mathematics

Euclid's *Elements* forms a natural breakpoint in Greek mathematical history; almost everything that came before the *Elements* was eventually swallowed by it. Pre-Euclidean mathematics is therefore largely a discussion of the origins of some of the ideas in Euclid.

Thales of Miletus (c.624–546 BC)

Often thought of as the first western scientist, Thales is also important in mathematics.

- Olive Trader based in Miletus, a city-state in Anatolia.
- Contact with Babylonian traders/scholars probably led to his learning some geometry and an attempt to organize his discoveries.
- Stated some of the first abstract propositions: in particular,
  - The angles at the base of an isosceles triangle are equal.
  - Any circle is bisected by its diameter.
  - A triangle inscribed in a semi-circle is right-angled (still known as Thales' Theorem).



- Proofs not forthcoming. The major development was the stating of *abstract* general principles. Thales' propositions concern *all* triangles, circles, etc. The Babylonians and Egyptians were merely observed to use certain results in calculations and gave no indication that they appreciated the general nature of their results.
- Mathematical reasoning, if he conducted such, was almost certainly visual. For example, by 425 BC, Socrates could describe how to halve/double the area of a square by joining the midpoints of edges.
- Thales arguably more important to the history of reasoning: offered arguments/ discussions concerning the 'stuff' of which the universe is made.

Pythagoras of Samos c.572–497 BC

- Much travelled (Egypt, Asia, Babylon, Italy) though his story was probably over-emphasised after his death. Eventually settled in Croton (southeast Italy) where he founded a school/cult, persisting over 100 years after his death. Mathematical results/developments came from the group collectively.
- More of a mystic/philosopher than a mathematician. Core belief that number is fundamental to nature. Motto: "All is number". Emphasised form, pattern, proportion.
- Pythagoreans essentially practiced a mini-religion (they were vegetarians, belived in the transmigration of souls, etc.).
- The following quote[1] helps give a flavor of the Pythagorean way of life.

> After a testing period and after rigorous selection, the initiates of this order were allowed to hear the voice of the Master [Pythagoras] behind a curtain; but only after some years, when their souls had been further purified by music and by living in purity in accordance with the regulations, were they allowed to see him. This purification and the initiation into the mysteries of harmony and of numbers would enable the soul to approach [become] the Divine and thus escape the circular chain of re-births.

Several famous results are attributable to the Pythagoreans. They were particularly interested in musical harmony and the relationship of such to number. For instance, they related intervals in music to the ratios of lengths of vibrating strings:

- Identical strings whose lengths are in the ratio 2:1 vibrate an *octave* apart.
- A *perfect fifth* corresponds to the ratio 3:2.
- A *perfect fourth* corresponds to the ratio 4:3.

Using such intervals to tune musical instruments (in particular pianos) is still known as *Pythagorean tuning.*

Theorems 21–34 in Book IX of Euclid's *Elements* are Pythagorean in origin:

Theorem (IX.21)**.** *A sum of even numbers is even.*

Theorem (IX.27)**.** *Odd less odd is even.*

The Pythagoreans studied perfect numbers: equal to the sum of their proper divisors (e.g. $6 = 1+2+3$). They seem to have observed the following, though it is not known if they had a proof.

Theorem (IX.36)**.** *If* $2^n - 1$ *is prime then* $2^{n-1}(2^n - 1)$ *is perfect.*

They also considered square and triangular numbers and tried to express geometric shapes as numbers, all in service of their belief that all matter could be formed from the combination of basic shapes. Cultish the Pythagoreans may have been, but they discovered many things and certainly had lofty goals!

[1] Van der Waerden, *Science Awakening* pp 92–93

Length, Number, Incommensurability and Pythagoras' Theorem

Our modern notion of continuity facilitates tight relationship between length and number: any object can be measured with respect to any fixed length. For instance, we're happy stating that the diagonal of a square is $\sqrt{2}$ times the side. To the Greeks and other ancient cultures, the only numbers were *positive integers.* Appreciating the distinction between length and number is crucial to understanding several of the major ideas of Greek mathematics. In particular, it helps explain the primacy of Geometry in their mathematics: *lengths* are real things that the Greeks wanted to compare using *numbers*.

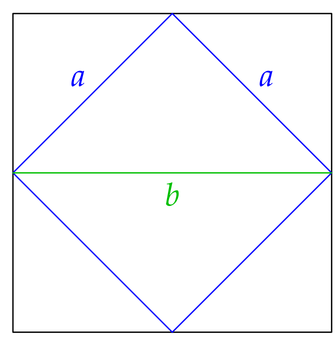
A core Pythagorean belief was that of *commensurability*: given two lengths, there exists a sub-length dividing exactly into both. One could then describe the relationship between lengths with a ratio. For instance, if the longer length contained three of the sub-lengths and the smaller two, this could be expressed with the ratio $3:2$. In modern language,

$$\forall x, y \in \mathbb{R}^+,\ \exists k, l \in \mathbb{N},\ r \in \mathbb{R}^+ \text{ such that } x = kr \text{ and } y = lr$$

This is complete nonsense for it insists that every ratio of real numbers is *rational!*
The supposition of commensurability clearly fits with the Pythagoreans' mystical emphasis on the perfection of number. The discovery that it was false produced something of a crisis. A (possibly) apocryphal story suggests that a disciple named Hippasus (c.500 BC) was set adrift at sea as punishment for revealing it. Nevertheless, it is generally accepted that the Pythagoreans provided the first evidence of the existence of irrational numbers in the form of incommensurable lengths.

By 340 BC, Aristotle was happy to state that incommensurable lengths exist.

Theorem (Aristotle)**.** *If the diagonal and side of a square are commensurable, then odd numbers equal even numbers.*



Inferred proof—original unknown. Consider Socrates' doubled-square from earlier. Label the sides of the blue square a and the diagonal b where a and b are integers denoting multiples of the common sub-length (the existence of a common sub-length is the hypothesis!).

We may assume that at least one of a or b is odd, for otherwise there exists a larger common sub-length. However the larger square (side b) is twice the smaller (side a) so b is even and the square number b^2 is divisible by four. But then a is also even.

Whichever of a, b was odd is also even: contradiction! ■

Note the similarity of this argument[2] to the standard modern proof of the irrationality of $\sqrt{2}$

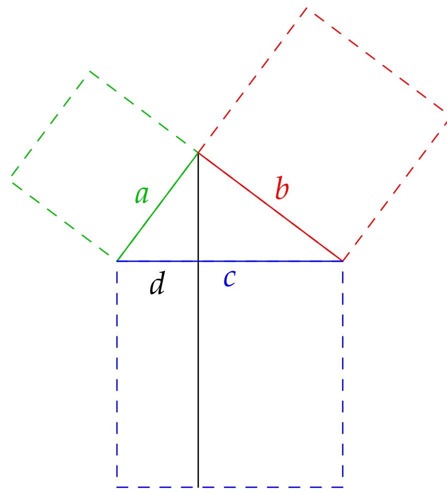
[2] For those with musical training, a similar argument shows that the Pythagorean notion of perfect fifths in music is also flawed. The *cycle of fifths* is a musical principal stating that an ascension through twelve perfect fifths takes you through every note in the standard chromatic scale, finishing seven octaves above where you start. This is essentially a claim that $2^7 = \left(\frac{3}{2}\right)^{12} \iff 2^{19} = 3^{12}$: palpable nonsense!

While there is no evidence that the Pythagoreans ever provided a correct proof of their famous Theorem, one argument possibly attributable to the Pythagoreans used the idea of commensurability.

'Proof' of Pythagoras' Theorem. Label the right triangle a, b, c where c is the hypotenuse and drop the altitude to the hypotenuse. Let d be the length of the a-side of the hypotenuse. Similar triangles tell us that

$$a : d = c : a \implies a^2 : ad = cd : ad \implies a^2 = cd \qquad (*)$$

Thus the square on a has the same area as the rectangle below the d-side of the hypotenuse. Repeat the calculation on the other side to obtain $b^2 = c(c - d)$. Now sum these for the proof. ■



In the language of the Pythagoreans, the only acceptable *numbers* were integers, so the symbols a, b, c, d in were the *integer multiples* of an assumed common sub-length. This restriction completely destroys the generality of the proof.

It is clear from the organization of Book I of Euclid's *Elements* that one of its primary goals was to provide a rigorous proof of Pythagoras' Theorem which did not depend on the flawed notion of commensurability. With our modern understanding of real numbers, there is nothing wrong with the above argument. We have fully adapted to the idea that length and number are freely interchangeable due to the completeness of the real numbers: ancient mathematicians had no such knowledge.

Zeno of Elea c.450 BC

Zeno might be called the patron saint of devil's advocates. His fame comes from his suggestion of several ingenious arguments/paradoxes involving the infinite and the infinitesimal. Here are perhaps the two most famous of these.

- Achilles and Tortoise: Achilles starts a race behind a Tortoise. After a given time t_1, Achilles reaches the Tortoise's starting position, but the Tortoise has moved on. After another time interval t_2, Achilles reaches the Tortoise's second position: again the Tortoise has departed. In this manner Achilles spends an infinite sum of time intervals $t_1 + t_2 + t_3 + \cdots$ in the chase. Zeno's paradoxical conclusion: Achilles never catches the Tortoise.
  The problem was that Zeno refused to accept that the total duration could be finite, even though it be split into infinitely many subintervals of time. The resolution of this paradox is at the heart of the modern notion of infinite series.

- Arrow paradox: An arrow is shot from a bow. At any given instant the arrow doesn't move. If time is made up of instants, then the arrow never moves.
  This time Zeno is debating the the idea that a finite time period can be considered as a sum of infinitessimal instants. Again the problem is one of limits and infinite sums.

Zeno's paradoxes have stimulated philosophers for thousands of years, continuing well into the 18th century as the theory of calculus was fleshed out. We shall revisit this controversy later. For the present, it is enough to consider how radical the fundamental ideas of calculus are: to measure an area one essentially slices it into infinitely many, infinitesimally thin strips before summing them up. Try selling this idea to someone who has never studied calculus!