## Ancient Trigonometry & Astronomy

Astronomy was hugely important to ancient cultures and became one of the most important drivers of mathematical development, particularly *Trigonometry* (literally *triangle-measure*). Calendars were often based on the phases of the moon (the origin of the word *month*) and the seasons. Tracking and predicting heavenly behavior was also a large part of religious practice.[1] As people travelled further, astronomy became increasingly important for navigation.

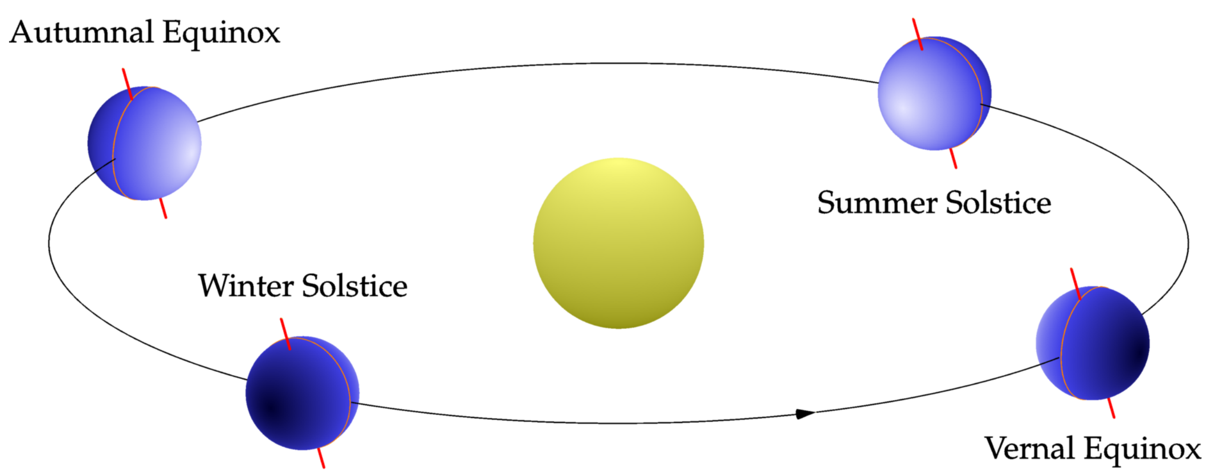
**Background and concepts** Amongst many others, the ancient Egyptians, Babylonians, Greeks, Indians and Chinese made rigorous measurements of the sky. Modern astronomical co-ordinates come to us courtesy of the Babylonians. Their astrological system has its roots around 2000 BC, but by 7-800 BC they had settled on a structure which is more or less familiar to modern astrologers.[2] To understand astronomical measurement, one first has to appreciate the *seasons.*

Seasonal variation exists because the Earth's axis is tilted approximately 23° from the vertical with respect to the *ecliptic* (Sun-Earth orbital plane). Summer, in a given hemisphere, is when the the Earth's axis is tilted towards the Sun, resulting in more sunlight and longer days. Astronomically, the seasons are determined by four dates (with respect to the northern hemisphere):

*Summer/Winter Solstice* (c.21st June/December) The north pole is maximally tilted towards/away from the sun. *Solstice* comes from a Latin (Roman) term meaning 'sun stationary.' If one measures the maximum elevation of the sun each day (the timing is the local definition of *noon*), then the summer solstice represents the date on which the sun is highest in the sky.

*Vernal/Autumnal Equinox* (c. 21st March/September) The tilt of the Earth's axis lies in the plane perpendicular to the Sun-Earth radius. *Equinox* means *equal night*: day and night both last approximately 12 hours since the Earth's axis passes through the day-night boundary.

Accurate prediction and observation of these dates was crucial for orienting the year and referencing other heavenly measurements. The picture shows the orientation of the ecliptic, the Earth's axis and the day-night boundary.




[1] For example, ancient Egyptian religion included the belief that the region around the pole star was the location of their heaven. Pyramids included narrow shafts pointing from the burial chamber to this region of the sky so that the deceased could 'ascend to the stars.'

[2] Thankfully modern astrology has got past divination using goat livers...

Measurements can now be conducted relative to this set-up.

- The *fixed stars* form the background with respect to which everything else is measured. The *ecliptic* is viewed as the Sun's apparent path over the year set against the fixed stars. Planets (*wandering stars*) were observed to move against this background.
- The year being roughly 360 days, one degree of measure along the ecliptic corresponds approximately to the Sun's apparent daily motion. *Celestial longitude* was measured from zero to 360° around the ecliptic with zero fixed at the vernal equinox.
- The ecliptic was divided into twelve segments each of 30°. These described the Babylonian zodiac: in modern terms Aries is 0–30° (vernal equinox $\to$ April 21st), Taurus is 30–60°, etc.
- *Celestial latitude* was measured in degrees north or south of the ecliptic, so that the Sun always had latitude zero.

Having acquired Babylonian knowledge through trade and the conquests of Alexander, the Greeks co-opted this system, but using Greek characters instead of the Babylonian $\vee$ and $\lessdot$. The base-60 degrees-minutes-seconds system persists in navigation/astronomy to this day.[3]

Early Greek analysis of the heavens was based on several assumptions.

- Spheres and circles are perfect, matching the gods' perfect design of the universe. The earth is a sphere and the 'fixed stars' (constellations) lie on larger 'celestial sphere.' All explanations of heavenly motion must rely on spheres and circles rotating at a constant rate.
- The Earth is stationary, so the celestial sphere rotates around the Earth once per day.
- The planets were assumed to lie on concentric spherical shells also centered on the Earth.

Two major contradictions were apparent when the above philosophy was tested by observation.

- The apparent brightness of heavenly bodies is not constant. In particular, the brightness of the planets is highly variable.
- Retrograde motion: planets mostly follow the East-West motion of the heavens, however they are occasionally seen to slow down and reverse course.

If all heavenly bodies are simply moving in circles centered on the Earth at a constant speed, then how can the above be explained? The attempt to produce accurate models while 'saving the phenomena' of spherical and circular motion led to the development of new mathematics. Our discussion begins with the approaches of two previously-encountered Greek mathematicians.

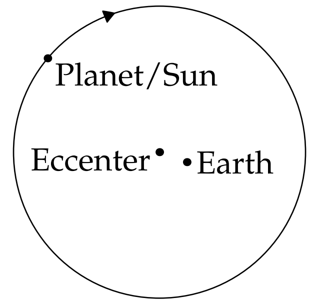
**Eudoxus of Knidos (c.390–340 BC)**

- Developed a multiple-sphere model where each planet or the sun lay on a sphere whose poles were attached to the sphere outside it, with the outermost containing the fixed stars.
- The motion generated by nested spheres becomes highly complex.[4] It is capable of describing retrograde motion, but not the variable brightness of stars and planets.

[3] In modern times longitude is measured in *hours-minutes-seconds* with 24 hours = 360°, and latitude/longitude (declination/right ascension) are measured with respect to the Earth's equatorial plane rather than the ecliptic. These *equatorial co-ordinates* were introduced by Hipparchus of Nicaea (below).

[4] The link directs to a rather nice flash animation of Eudoxus' model.

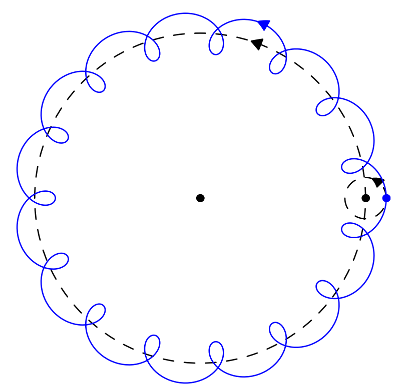
**Apollonius of Perga ($2^{nd}$/$3^{rd}$C BC)**




Apollonius proposed two solutions. In his *eccenter* model, the sun is assumed to orbit a circle (the deferent) whose center is *not* the Earth. This rather simply addressed the problem of variable brightness.

The criticism is obvious: *why?* What is the philosophical justification for the eccenter? Eudoxus' model may have been complex and essentially impossible to compute with, but it was more in line with the assumptions of spherical motion. Apollonius, however, wasn't done...

His second approach modelled plantary motion using *epicycles*: video link.



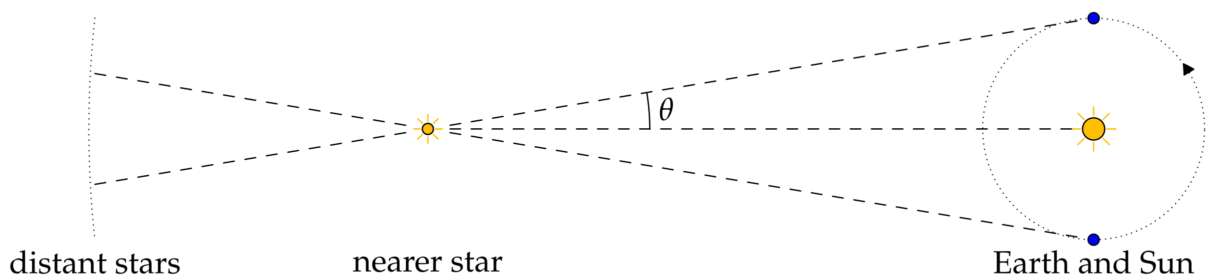
An epicycle is a small circle moving around a larger: you'll be familiar with these if you've played with the toy *Spirograph.* The observer at the center will see the apparent brightness change, and potentially observe retrograde motion. In modern language, the motion is parametrized by the vector-valued function

$$\mathbf{x}(t) = R\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix} + r\begin{pmatrix}\cos\psi t\\ \sin\psi t\end{pmatrix}$$

where $R, r, \omega, \psi$ are the radii and circular frequencies (rad/s) of the two circles.

Combining these models gave Apollonius the ability to describe quite complex motion. Calculation was difficult however, requiring finding lengths of chords of circles from a given angle, and vice versa. It is from this requirement that the earliest notions of trigonometry arise.

One might ask why the Greeks didn't make the 'obvious' fix and place the sun at the center of the cosmos. In fact Aristarchus of Samos (c.310–230 BC) did precisely this, suggesting that the fixed stars were really just other suns at exceptional distance. However, the great thinkers of the time (Plato, Aristotle, etc.) had a very strong objection to Aristarchus approach: *parallax.*




If the Earth moves round the sun, and the fixed stars are really independent objects, then the position of a nearer star should appear to change throughout the year. The angle $\theta$ in the picture is the *parallax* of the nearer star. Unfortunately for Aristarchus, the Greeks were incapable of observing any parallax.[5] It wasn't until Copernicus and Kepler in 15-1600's that *heliocentric* models began to be taken seriously.

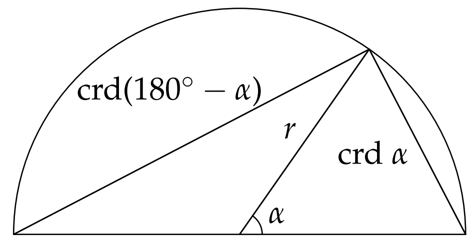
[5]The astronomical unit of one *parsec* is the distance of a star exhibiting one arc-second ($\frac{1}{3600}°$) of parallax, roughly 3.3 light-years or $3 \times 10^{13}$ km, an unimaginable distance to anyone before the scientific revolution. The nearest star to the Sun is Proxima Centauri at 4.2 light years $= 0.77$ parsecs: is it any wonder the Greeks rejected the hypothesis?!

**Hipparchus of Nicaea (c.190–120 BC)**

Hipparchus was one of the pre-emminent Greek astronomers. He made great use of Babylonian eclipse data to fit Apollonius' eccenter and epicycle models to the observed motion of the moon. As part of this work, he needed to be able to compute chords of circles with some accuracy: his *chord tables* are acknowledged as the earliest tables of trigonometric values.

In an imitation of Hipparchus' approach, we define a function crd which returns the length of the chord in a given circle subtended by a given angle. Translated to modern language,

$$\operatorname{crd}\alpha = 2r\sin\frac{\alpha}{2}$$




Hipparchus chose a circle with circumference 360° (in fact he used $60 \cdot 360 = 21600$ arcminutes): the result is that $r = \frac{21600}{2\pi} \approx 57,18;$, sixty times the number of degrees per radian.[6] His chord table was constructed starting with the obvious:

$$\operatorname{crd} 60^\circ = r = 57,18; \qquad \operatorname{crd} 90^\circ = \sqrt{2}r = 81,2;$$

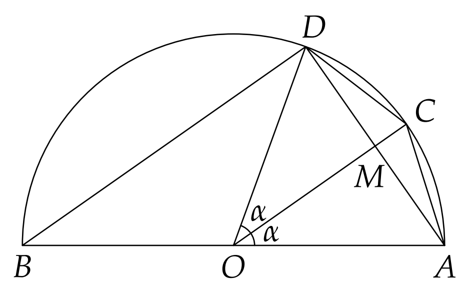
Pythagoras' Theorem was used to obtain chords for angles $180^\circ - \alpha$. In modern language

$$\operatorname{crd}(180^\circ - \alpha) = \sqrt{(2r)^2 - (\operatorname{crd}\alpha)^2} = 2r\sqrt{1 - \sin^2(\alpha/2)} = 2r\cos\frac{\alpha}{2}$$

Pythagoras' was again used to halve and double angles in an approach analogous to Archimedes' quadrature of the circle. We rewrite the argument in this language.

In the picture, $|DB| = \operatorname{crd}(180 - 2\alpha)$. Since $\angle BDA = 90^\circ$ and $M$ is the midpoint of $\overline{AD}$, we see that $\triangle OMD$ is right-angled. Thus

$$|OM| = \frac{1}{2}|BD| = \frac{1}{2}\operatorname{crd}(180 - 2\alpha)$$




Now apply Pythagoras' to $\triangle CMD$:

$$\begin{aligned}
(\operatorname{crd}\alpha)^2 &= \left(\frac{1}{2}\operatorname{crd}2\alpha\right)^2 + \left(r - \frac{1}{2}\operatorname{crd}(180 - 2\alpha)\right)^2 \\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180 - 2\alpha) + \frac{1}{4}\operatorname{crd}(180 - 2\alpha)^2 \\
&= \frac{1}{4}(\operatorname{crd}2\alpha)^2 + r^2 - r\operatorname{crd}(180 - 2\alpha) + \frac{1}{4}(4r^2 - (\operatorname{crd}2\alpha)^2) \\
&= 2r^2 - r\operatorname{crd}(180 - 2\alpha)
\end{aligned}$$

In modern notation this is one of the double-angle trigonometric identities!

$$4r^2\sin^2\frac{\alpha}{2} = 2r^2 - 2r^2\cos\alpha \iff \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

[6] One radian is defined as the angle subtended by an arc equal in length to the radius of a circle. Hipparchus essentially does this in reverse: if the circumference is fixed so that *degree* now measures both subtended angle *and* circumferential distance, then the radius of the circle is fixed.

**Example** To calculate $\operatorname{crd} 30°$, we start with $\operatorname{crd} 60° = r$. Then

$$\begin{aligned} &\operatorname{crd} 120° = \sqrt{4r^2 - r^2} = \sqrt{3}r \\ \implies &\operatorname{crd} 30° = \sqrt{2r^2 - r\operatorname{crd}(180° - 60°)} = \sqrt{2r^2 - \sqrt{3}r^2} = \sqrt{2-\sqrt{3}}r \end{aligned}$$

In modern language this yields an exact value for $\sin 15°$:

$$\operatorname{crd} 30° = 2r \sin 15° \implies \sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$$

Continuing this process, we obtain $\operatorname{crd} 150° = r\sqrt{2+\sqrt{3}}$, whence

$$(\operatorname{crd} 15°)^2 = 2r^2 - r\operatorname{crd} 150° = (2 - \sqrt{2+\sqrt{3}})r \implies \operatorname{crd} 15° = \sqrt{2-\sqrt{2+\sqrt{3}}}r$$

Again translating: $\sin 7.5° = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}$.

Using this approach, Hipparchus computed the chord of each of the angles $7.5°$, $15°, \ldots, 172.5°$, in steps of $7.5°$. Of course everything was an estimate since he had to rely on approximations for square-roots.

All Hipparchus' original work is now lost. We primarily know of his work by reference. In particular, the above method of chords is probably due to Hipparchus, although we see it first in the work of. . . [7]

**Claudius Ptolemy (c.100–170 AD)**

- Produced the *Mathematica Syntaxis* around 150 AD, better known as the *Almagest*: the latter term is derived from the Arabic *al-mageisti*, meaning *great work* and reflects the fact that renaissance Europe inherited the text from the Islamic world.
- The *Almagest* is essentially a textbook on geocentric cosmology. It shows how to compute the motions of the moon, sun and planets, describing lunar parallax, eclipses, the constellations, etc. It contains our best evidence as to the accomplisments of Hipparchus and describes his calculations. The text formed the basis of Western/Islamic astronomical theory through to the 1600's: it is essentially the *Elements* of Astronomy.
- It is important to note that, following the Pythagoreans and Plato, Ptolemy did not find it necessary to seek physical causes for the motion of the planets. The mathematics and the description of what one would see was enough.
- From a trigonometric point of view, the most important feature of the *Almagest* was its dramatic improvement on the chord tables of Hipparchus, and the inclusion of elementary *spherical trigonometry*, probably courtesy of Menelaus (c.100 AD).
- Ptolemy lived in Alexandria: while the name Ptolemy is Greek, his other name Claudius is Roman, reflecting the changing cultural situation.

[7] The essential computation of side length for doubling the number of sides of an inscribed polygon is precisely that seen in the work of Archimedes which came earlier. You should compare the approaches.

**Ptolemy's Calculations** Ptolemy computed more chords and with greater accuracy than Hipparchus. To assist with this, he made one critical change and a number of improvements to the method.

- Ptolemy started with $r = 60$ rather than $r = 57;18$. This made for easier calculations, in particular $\operatorname{crd} 60^\circ = 60$. Ptolemy then used what was probably Hipparchus' method to halve/double angles and compute chords of supplementary angles:

$$\operatorname{crd}^2 \alpha = 2r^2 - r \operatorname{crd}(180^\circ - 2\alpha) = 60\big(120 - \operatorname{crd}(180^\circ - 2\alpha)\big)$$
$$\operatorname{crd}(180^\circ - \alpha) = \sqrt{(2r)^2 - \operatorname{crd}^2 \alpha} = \sqrt{120^2 - \operatorname{crd}^2 \alpha}$$

  with square-roots approximated to the desired accuracy. For example,

$$\operatorname{crd}^2 30^\circ = 60(120 - \operatorname{crd} 120^\circ) = 60\left(120 - 60\sqrt{3}\right) = 60^2(2 - \sqrt{3})$$
$$\implies \operatorname{crd} 30^\circ = 60\sqrt{2 - \sqrt{3}} \approx 31;3,30$$

- Ptolemy had more initial data that Hipparchus: in particular,

$$\operatorname{crd} 60^\circ = 60, \quad \operatorname{crd} 90^\circ = 60\sqrt{2}, \quad \operatorname{crd} 36^\circ = 30(\sqrt{5} - 1), \quad \operatorname{crd} 72^\circ = 30\sqrt{10 - 2\sqrt{5}}$$

  The first two are easy; we'll explain how he knew the latter in a moment.

- Ptolemy could add and subtract angles using what is now known as Ptolemy's Theorem (see below). He computed $\operatorname{crd} 12^\circ = \operatorname{crd}(72^\circ - 60^\circ)$, then halved this for angles of 6°, 3°, 1.5°, and 0.75°. Chords for all integer multiples of 1.5° could then be computed using addition formula.

- For finer detail, Ptolemy used interpolation. His essential observation was that

$$\alpha < \beta \implies \frac{\operatorname{crd} \beta}{\operatorname{crd} \alpha} < \frac{\beta}{\alpha}$$

  In modern language $\frac{\sin \hat{\beta}}{\beta} < \frac{\sin \hat{\alpha}}{\hat{\alpha}}$ so that $\frac{\sin x}{x}$ increases to 1 as $x \to 0^+$. This allowed him to compute the chord for every integer multiple of $\frac{1}{2}^\circ$ to the incredible accuracy of 1 part in 3600 (two sexagesimal places).

- For approximating between half-degrees, he supplied a second table of values indicating how much should be added for each arcminute ($\frac{1}{60}^\circ$). For example, the second line of Ptolemy's table reads

  1°    1;2,50    ;1,2,50

  meaning first that $\operatorname{crd} 1^\circ = 1;2,50$ to two sexagesimal places.[8] The second entry says, for example that

$$\operatorname{crd}(1^\circ 5') \approx 1;2,50 + 5(;1,2,50) = 1;8,4,10 \approx 1;8;4$$

  It is believed that Ptolemy computed his half-angle chords to an accuracy of *five sexagesimal places* in order to obtain his arcminute approximations.

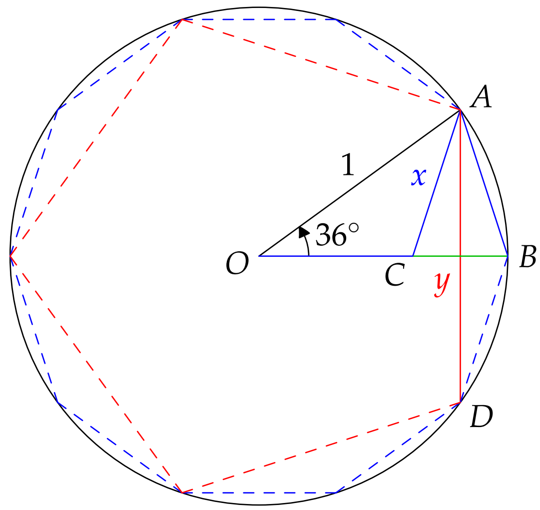
[8]This is $1 + \frac{2}{60} + \frac{50}{60^2} = 1.0472222\ldots = 120 \sin \frac{1.00003625\ldots}{2}^\circ$, an already phenomenal level of accuracy.

**How did Ptolemy know the values of** crd 36° **and** crd 72°**?** Everything necessary is in the *Elements.*

**Theorem.** 1. *(Thm XIII.9) In a circle, the sides of a regular inscribed hexagon and decagon are in the golden ratio (this ratio is* 60 : crd 36° *in Ptolemy).*

2. *(Thm XIII.10) In any circle, the square on an inscribed pentagon equals the sum of the squares on the inscribed hexagon and decagon.*

Purely Euclidean proofs are too difficult for us, so here is a way to see things in modern notation.




1. Let $\overline{AB} = x$ be the side of a regular decagon inscribed in a unit circle with center $O$.

   $\triangle OAB$ is isosceles with angles $36°$, $72°$, $72°$.

   Let $C$ lie on $\overline{OB}$ such that $\overline{AC} = x$.

   It is easy to check (count angles!) that $\triangle OAB$ and $\triangle ABC$ are similar and that $\overline{OC} = x$.

   Clearly $x = \frac{1-x}{x}$, whence $x = \frac{\sqrt{5}-1}{2}$.

   In a circle of radius 60, this gives the exact value

   $$\text{crd}\, 36° = 60x = 30(\sqrt{5}-1)$$

2. Now let $\overline{AD} = y$ be the side of a regular pentagon inscribed in the same circle. Applying Pythagoras' Theorem, we see that

   $$\left(\frac{y}{2}\right)^2 + \left(\frac{1-x}{2}\right)^2 = x^2$$

   Using the fact that $x^2 = 1 - x$, we may quickly multiply out to obtain Euclid's result

   $$y^2 = 1^2 + x^2$$

   from which we obtain the exact value

   $$\text{crd}\, 72° = 60y = 30\sqrt{10 - 2\sqrt{5}}$$

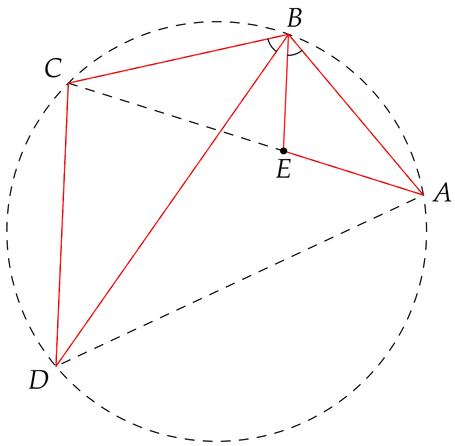
While these observations were geometrically precise, Ptolemy used sexagesmial approxomations to square-roots to obtain

$$\text{crd}\, 36° = 37;4,55 \qquad \text{crd}\, 72° = 70;32,3$$

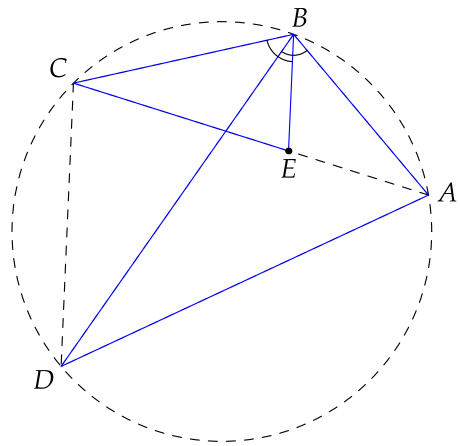
While these are the values stated in his tables, he must have used a far higher degree of accuracy in order to obtain similarly accurate values for other chords.

**Theorem** (Ptolemy's Theorem)**.** *Suppose a quadrilateral is inscribed in a circle. Then the product of the diagonals equals the sum of the products of the opposite sides.*[9]

*Proof.* Choose a point $E$ on $\overline{AC}$ such that $\angle ABE \cong \angle DBC$. Then $\angle ABD \cong \angle EBC$. We obtain two pairs of similar triangles:[10]




$\triangle ABE \sim \triangle DBC$




$\triangle ABD \sim \triangle EBC$

The proof follows immediately: compute $\frac{|AE|}{|CD|} = \frac{|AB|}{|BD|}$ and $\frac{|CE|}{|AD|} = \frac{|BC|}{|BD|}$, whence

$$|AC|\,|BD| = (|AE| + |CE|)\,|BD| = |AB|\,|CD| + |AD|\,|BC|$$

which gives the result. ∎

A special case of Ptolemy's Theorem allowed him to find chords of sums and differences of angles: these are precisely the multiple-angle formulæ from modern trigonometry.

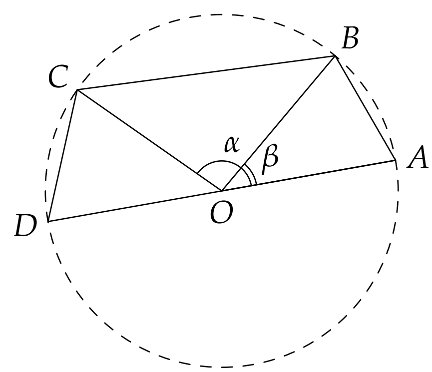
**Corollary** (Multiple Angle Formula)**.** *If $\alpha > \beta$, then*

$$120\,\mathrm{crd}(\alpha - \beta) = \mathrm{crd}\,\alpha\,\mathrm{crd}(180^\circ - \beta) - \mathrm{crd}\,\beta\,\mathrm{crd}(180^\circ - \alpha)$$

*In modern language, divide out by $120^2$ to obtain*

$$\sin\frac{\alpha - \beta}{2} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\alpha}{2}$$

*Proof.* Suppose that $|AD| = 120$ is a diameter of the drawn circle. Ptolemy's Theorem says




$$|AC|\,|BD| = |AB|\,|CD| + |AD|\,|BC|$$

which is simply

$$\mathrm{crd}\,\alpha\,\mathrm{crd}(180^\circ - \beta) = \mathrm{crd}\,\beta\,\mathrm{crd}(180^\circ - \alpha) + 120\,\mathrm{crd}(\alpha - \beta)$$ ∎

[9]There is some argument as to whether this result is in Euclid's *Elements.* Book VI of Euclid traditionally contains 33 propositions, however some editions append four extra results as corollaries, Ptolemy's Theorem being the last of these (Thm VI.D). It is generally considered that the result itself predates Ptolemy.

[10]$\angle BAE \cong \angle BDC$ since both are inscribed angles of the same arc $\overline{BC}$.

Similar expressions for $\operatorname{crd}(\alpha+\beta)$ and $\operatorname{crd}(180^\circ-(\alpha\pm\beta))$ were also obtained which essentially recover the modern expressions for $\sin(\alpha\pm\beta)$ and $\cos(\alpha\pm\beta)$. It is now trivial to obtain the expression $\sin 2\alpha=\sin(\alpha+\alpha)=2\sin\alpha\cos\alpha$, etc.

**Examples** Here is how Ptolemy might have calculated $\operatorname{crd} 42^\circ$. Let $\alpha=72^\circ$ and $\beta=30^\circ$, then

$$120\operatorname{crd}42^\circ=\operatorname{crd}72^\circ\operatorname{crd}150^\circ-\operatorname{crd}30^\circ\operatorname{crd}108^\circ$$

Since $\operatorname{crd}72^\circ=30\sqrt{10-2\sqrt{5}}$ is the side-length of a regular pentagon, and

$$\operatorname{crd}108^\circ=\operatorname{crd}(180^\circ-72^\circ)=\sqrt{120^2-\operatorname{crd}^2 72^\circ}=30\sqrt{6+2\sqrt{5}}$$

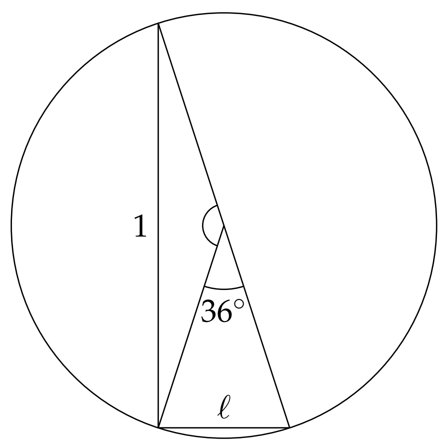
we see that

$$\begin{aligned}\operatorname{crd}42^\circ&=\frac{1}{120}\left(30\sqrt{10-2\sqrt{5}}\cdot 60\sqrt{2+\sqrt{3}}-60\sqrt{2-\sqrt{3}}\cdot 30\sqrt{6+2\sqrt{5}}\right)\\&=15\left(\sqrt{10-2\sqrt{5}}\cdot\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}\cdot\sqrt{6+2\sqrt{5}}\right)\approx 43;0,15\approx 43.0042\end{aligned}$$

Note all the square-root terms which had to be approximated: the construction of the chord-table was truly a gargantuan task, a task for which Ptolemy almost certainly had assistance.

The *Almagest* contained a huge number of examples to aid the reader in their own calculations. Here is one such.

At a particular point at noon, a stick of length 1 is placed in the ground. The angle of elevation of the sun is 72°. What is the length of its shadow?




To solve the problem, Ptolemy instructs the reader to draw a picture and use ratios. The lower isosceles triangle has base angles 72°, as required. We use modern notation: clearly

$$\begin{aligned}&1:\ell=\operatorname{crd}144^\circ:\operatorname{crd}36^\circ\\\Longrightarrow\ &\ell=\frac{\operatorname{crd}36^\circ}{\operatorname{crd}144^\circ}=\frac{30(\sqrt{5}-1)}{30\sqrt{10+2\sqrt{5}}}\approx 0.32491\end{aligned}$$

Note that this is precisely $\cot 72^\circ$, though Ptolemy had no such notion.
Further calculations and examples were far more complex!

It is worth noting that Ptolemy's system was resolutely *geocentric.* Similarly to how the fame of Euclid's *Elements* shaped 2000 years of mathematical principle, the massive impact of Ptolemy's *Almagest* and the reverence felt for ancient knowledge meant that it became increasingly difficult in medieval Europe to challenge geocentrism. By the time Galileo was advancing the Copernican theory in the 1600's, heliocentrism was essentially heresy.