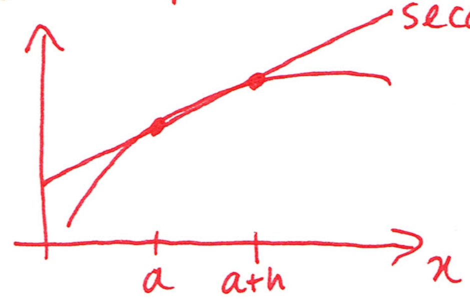
§2.1 Tangent line picture



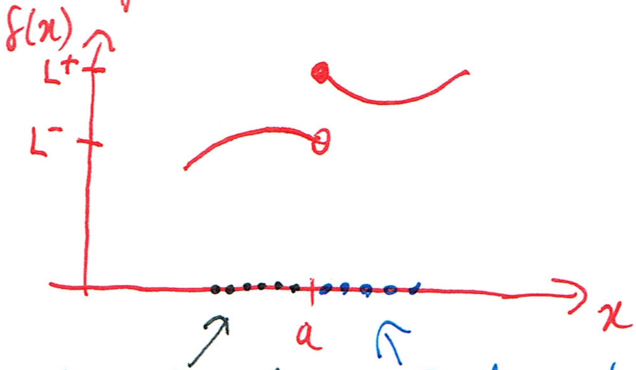
secant line approximates tangent line at $x=a$ if h is small

gradient of secant line

$$= \frac{f(a+h)-f(a)}{h}$$

Velocity: If $d=f(t)$ is the distance traveled by a particle (meters) at time t (sec), then the tangent line has slope = velocity of particle. (Units = m/s)

§2.2 Limits picture



$$\lim_{x\to a^-} f(x) = L^-$$

$$\lim_{x\to a^+} f(x) = L^+$$

Here $\lim_{x\to a} f(x) = \text{DNE}$

Feed in values of $x<a$, getting closer to a to find $\lim_{x\to a^-} f(x)$

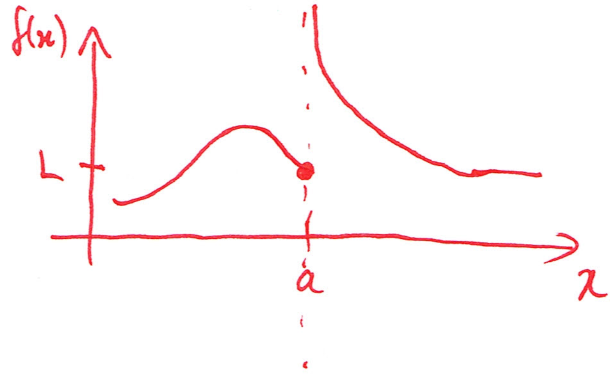
Feed in values of $x>a$, getting closer to a to find $\lim_{x\to a^+} f(x)$

Thm $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

- Vertical asymptote: $x=a$ if any of ~~six~~ three limits

$\lim_{x\to a} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ are ∞ or $-\infty$.

e.g.



$\lim_{x\to a^-} f(x) = L$

$\lim_{x\to a^+} f(x) = \infty$

so $x=a$ is vertical asymptote.

§2.3 Limit laws: Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

$\lim_{x\to a} cf(x) = cL$, $\lim_{x\to a} (f(x) \pm g(x)) = L \pm M$, $\lim_{x\to a} f(x)g(x) = LM$

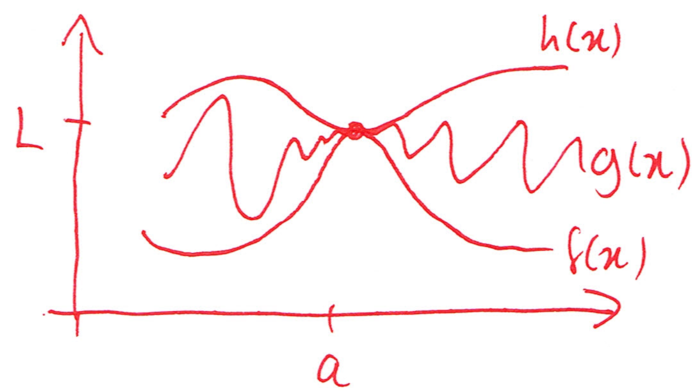
$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ (if $M \neq 0$), $\lim_{x\to a} x^n = a^n$, $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} = a^{1/n}$.

Rational functions: e.g. $\lim_{x\to 2^\pm} \frac{x^2-3x+2}{x^2-4x+4} = ?$

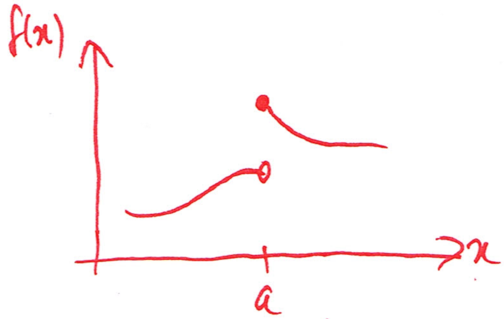
Rationalizing/conjugate method: e.g. $\lim_{x\to 4} \frac{\sqrt{x^2+9}-5}{x-4} = 4/5$.

Squeeze thm: Suppose $f(x) \le g(x) \le h(x)$ except perhaps at $x=a$ and suppose $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$.

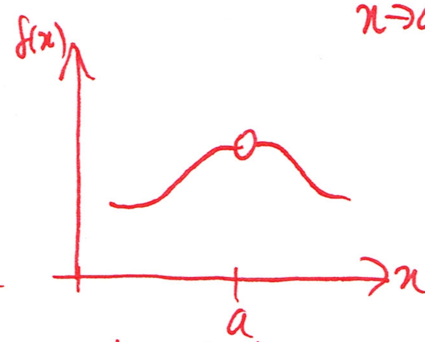
Then $\lim_{x\to a} h(x) = L$



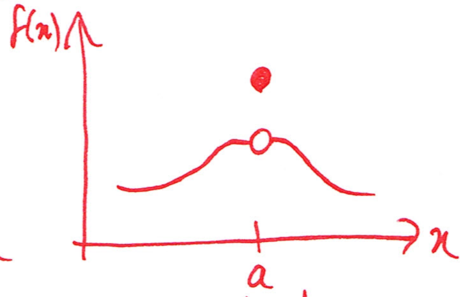
§2.5 Continuity: f cont at $x=a \iff \lim_{x\to a} f(x) = f(a)$



not cont at $x=a$:
$\lim_{x\to a} f(x) = \text{DNE}$



not cont at $x=a$:
$f(a) = \text{DNE}$

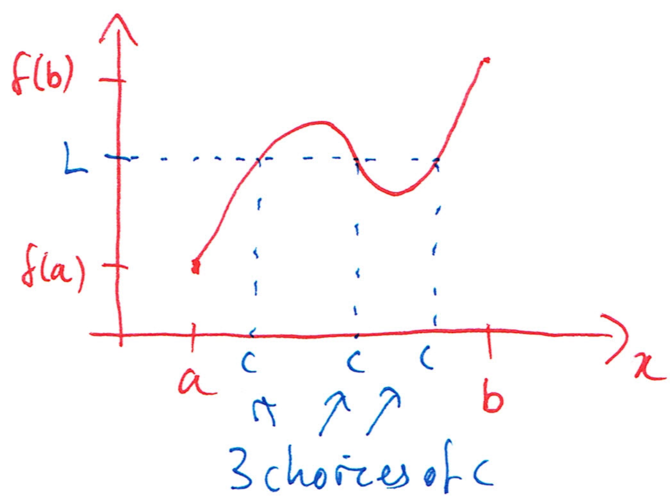


not cont at $x=a$:
$\lim_{x\to a} f(x) \neq f(a)$

At every point of their domains, polynomials, rational functions, powers, roots, exponentials, trigonometric functions, etc., are <u>all</u> continuous.

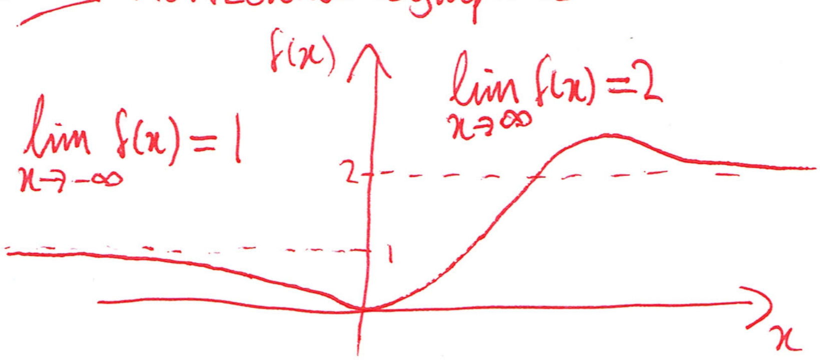
Compositions ($f\circ g$) and inverses (f^{-1}) of continuous functions are continuous.

Intermediate Value Theorem: Let f be continuous on $[a,b]$. If L is between $f(a)$ and $f(b)$, then there is at least one $c \in (a,b)$ such that $f(c) = L$.



- Show a root exists:
  e.g. $x^{17} + \cos x = 0$ has a solution $x \in (-1, 0)$
- Word problems...

§2.6 Horizontal asymptotes



- e^x and e^{-x}
- $\tan^{-1} x$
- Rational functions:
  e.g. $\dfrac{x^3+7x+1}{(2x+1)(3x+1)\,x} \xrightarrow[x\to\pm\infty]{} \dfrac{1}{6}$

Dealing with roots: $\sqrt[n]{x^n} = \begin{cases} x & \text{if } n \text{ odd} \\ |x| & \text{if } n \text{ even} \end{cases}$

i.e. $\sqrt{(-2)^2} = 2$.

So $$\lim_{x\to-\infty} \frac{\sqrt{x^2+7}+2}{x-1} \underset{\lim_{x\to-\infty}}{=} \frac{|x|\sqrt{1+7/x^2}+2}{x-1}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+7/x^2} + 2/|x|}{\frac{x}{|x|} - \frac{1}{|x|}} = \lim_{x\to-\infty} \frac{\sqrt{1+\frac{7}{x^2}} + 2/|x|}{-1 - 1/|x|},$$

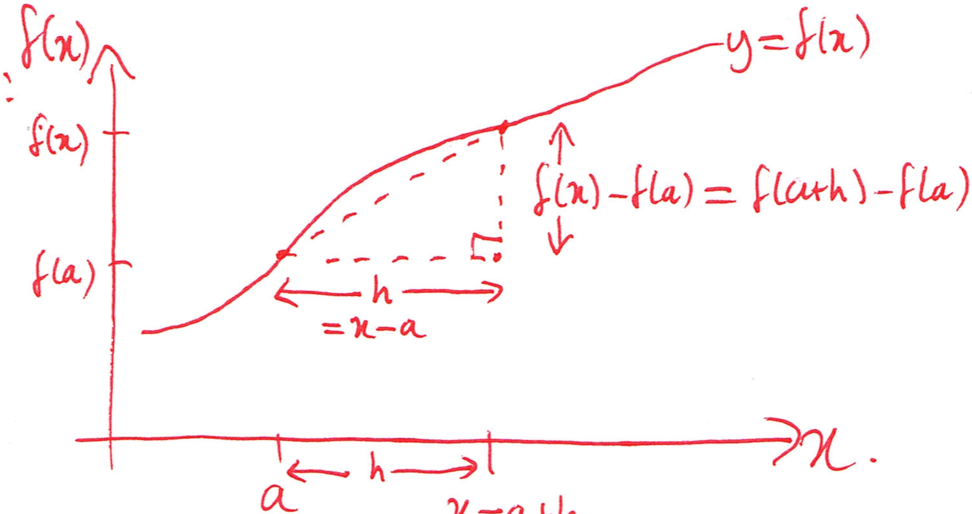
since $|x| = -x$ when $x < 0$.

$\therefore$ limit $= \dfrac{1+0}{-1-0} = -1$.

2.7 Derivative of $f(x)$: $$f'(a) = \lim_{h\to0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

__if__ limit exists.

Picture:



Velocity: $s'(a)$

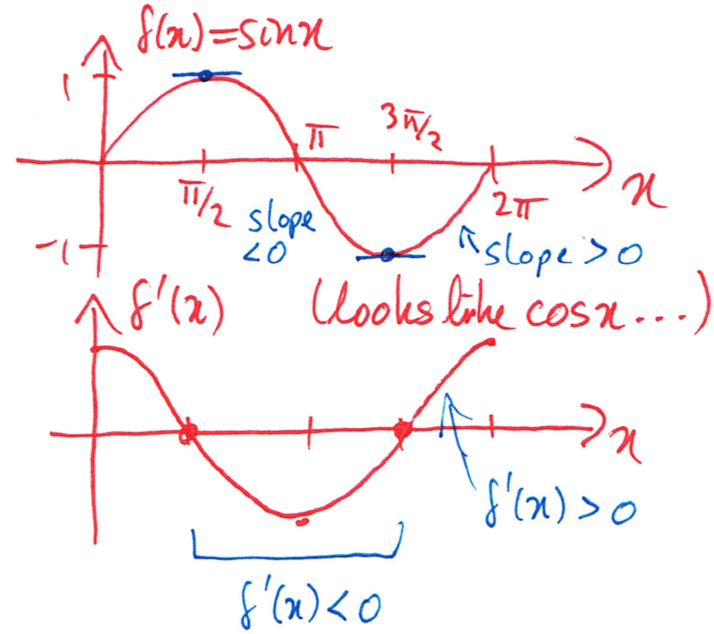
Average velocity over $a \le t \le b$: $v_{av} = \dfrac{s(b)-s(a)}{b-a}$

Leibniz notation: $\dfrac{df}{dx} = f'(x)$.

- Examples computing derivatives directly from limit definition.

2.8/ As a function $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, whenever limit exists.

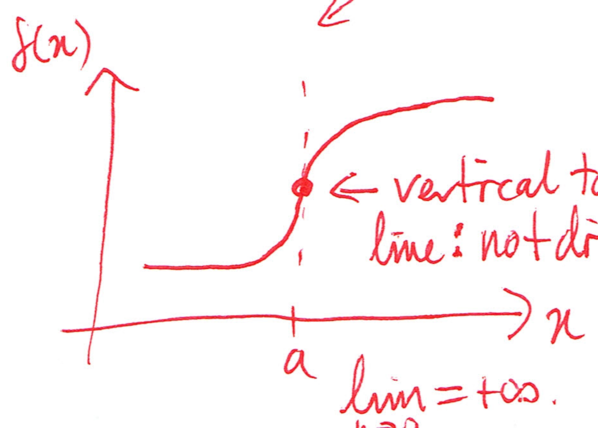
Visual differentiation:

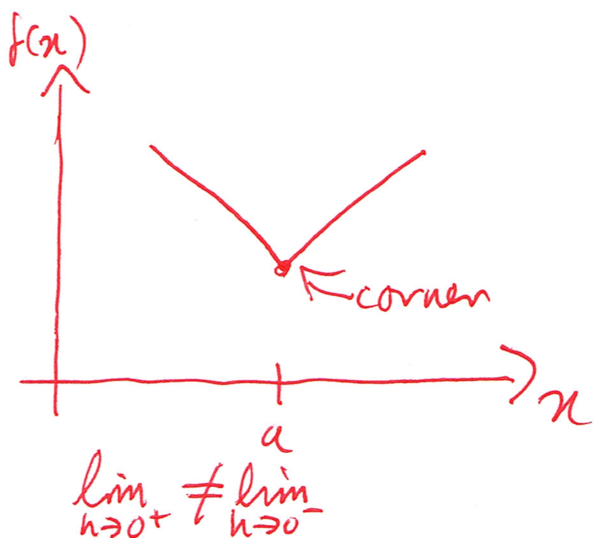


<u>Thm</u> If f is differentiable at $x=a$ then it is continuous at $x=a$.

How can a function <u>fail</u> to be differentiable?

1. If not continuous, or,
2. If $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = $ DNE or $\pm\infty$





Higher derivatives: $f''(x) = \dfrac{d^2 f}{dx^2}$, $f'''(x) = \dfrac{d^3 f}{dx^3}$, $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$

Alternate notation: $\dfrac{d^2 f}{dx^2} = \dfrac{d}{dx}\left(\dfrac{df}{dx}\right)$ helps with units.

e.g. if f has units meters & x units of seconds,

then $\dfrac{df}{dx} = f'(x)$ has units m/s, (speed)

$\dfrac{d^2 f}{dx^2} = f''(x)$ has units m/s^2 (acceleration)

$\dfrac{d^3 f}{dx^3} = f'''(x)$ has units m/s^3 "meters per second cubed"

- Compute $f(x)$ and $f''(x)$ for $f(x) = x|x| = \begin{cases} x^2 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x^2 & \text{if } x<0 \end{cases}$

Should find: $f'(x) = 2|x| = \begin{cases} 2x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -2x & \text{if } x<0 \end{cases}$

and $f''(x) = \begin{cases} 2 & \text{if } x>0 \\ -2 & \text{if } x<0 \end{cases}$

Check: $\lim\limits_{h\to 0} \dfrac{f'(0+h) - f'(0)}{h} = \text{DNE}$, so $f''(0)$ does not exist!