§2.1 Tangent line picture

secant line approximates tangent line at \( x = a \) if \( h \) is small.

Gradient of secant line:

\[
\frac{f(a+h) - f(a)}{h}
\]

Velocity: If \( d = f(t) \) is the distance traveled by a particle (meters) at time \( t \) (sec), then the tangent line has slope = velocity of particle. (Unit = m/sec)

§2.2 Limits picture

\[
\lim_{x \to a^-} f(x) = L^-
\]

\[
\lim_{x \to a^+} f(x) = L^+
\]

Here \( \lim_{x \to a} f(x) = \text{DNE} \)

Feed in values of \( x < a \), getting closer to a to find:

\[
\lim_{x \to a^-} f(x)
\]

Feed in values of \( x > a \), getting closer to a to find:

\[
\lim_{x \to a^+} f(x)
\]

\[\text{Then } \lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L\]
- Vertical asymptote: \( x = a \) if any of these limits
  \[
  \lim_{x \to a^-} f(x), \lim_{x \to a^+} f(x), \lim_{x \to a} f(x)
  \]
  are \( \infty \) or \( -\infty \).

  e.g. \( f(x) \)

  \[
  \lim_{x \to a^-} f(x) = L
  \]
  \[
  \lim_{x \to a^+} f(x) = \infty
  \]
  so \( x = a \) is vertical asymptote.

\[\text{§2.3 Limit laws: Suppose } \lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M. \text{ Then}
\]
\[
\lim_{x \to a} (f(x) \pm g(x)) = L \pm M, \quad \lim_{x \to a} f(x)g(x) = LM
\]
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{if } M \neq 0, \quad \lim_{x \to a} x^n = a^n, \quad \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} = a^{\frac{1}{n}}.
\]

Rational functions: e.g. \( \lim_{x \to 2^+} \frac{x^3 - 3x + 2}{x^2 - 4x + 4} = ? \)

Rationalizing/conjugate method: e.g. \( \lim_{x \to 4} \frac{\sqrt{x^2 + 9} - 5}{x - 4} = \frac{4}{5} \).

Squeeze Theorem: Suppose \( f(x) \leq g(x) \leq h(x) \) except perhaps at \( x = a \)

  and suppose \( \lim_{x \to a} f(x) = L = \lim_{x \to a} h(x) \).

  Then \( \lim_{x \to a} g(x) = L \).
§2.5 Continuity: \( f \) is continuous at \( x = a \) \( \iff \lim_{x \to a} f(x) = f(a) \)

- Not continuous at \( x = a \):
  - \( \lim_{x \to a} f(x) = \text{DNE} \)
  - \( f(a) = \text{DNE} \)
  - \( \lim_{x \to a} f(x) \neq f(a) \)

At every point of their domains, polynomials, rational functions, powers, roots, exponentials, trigonometric functions, etc., are all continuous.

Compositions \((f \circ g)\) and inverses \((f^{-1})\) of continuous functions are continuous.

Intermediate Value Theorem:

Let \( f \) be continuous on \([a, b]\). If \( L \) is between \( f(a) \) and \( f(b) \), then there is at least one \( c \in (a, b) \) such that \( f(c) = L \).

- Show a root exists:
  - e.g. \( x^3 + \cos x = 0 \) has a solution \( x \in (-1, 0) \)
- Word problems...

§2.6 Horizontal asymptotes

- \( e^x \) and \( e^{-x} \)
- \( \tan^{-1} x \)
- Rational functions:
  - e.g. \( \frac{x^3 + 3x + 1}{(2x + 1)(3x + 1)} x \to \infty \) \( \to \frac{1}{6} \)
Dealing with roots:
\[ n\sqrt[2]{x} = \begin{cases} x & \text{if odd} \\ 1x1 & \text{if even} \end{cases} \]
i.e. \[ n\sqrt[2]{(-2)^2} = 2. \]

So
\[
\lim_{x \to -\infty} \frac{n\sqrt{x^2 + 7} + 2}{x - 1} = \frac{1x1 \sqrt{1 + \frac{7}{x^2}} + 2}{x - 1}
\]
\[
= \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{7}{x^2}} + \frac{2}{1x1}}{\frac{x}{1x1} - \frac{1}{1x1}} = \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{7}{x^2}} + \frac{2}{1x1}}{-1 - \frac{1}{1x1}}
\]

Since \[ 1x1 = -x \text{ when } x < 0, \]
\[ \therefore \lim = \frac{1 + 0}{-1 - 0} = -1. \]

2.7 Derivative of \( f(x) \):
\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a},
\]
if limit exists.

Velocity:
\[ S'(a) \]
Average velocity over \( a \leq t \leq b \):
\[ \bar{v} = \frac{S(b) - S(a)}{b - a} \]
Leibniz notation: $\frac{df}{dx} = f'(x)$.

- Examples computing derivatives directly from limit definition.

$2.8$ As a function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, whenever limit exists.

Visual differentiation:

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Then if $f$ is differentiable at $x=a$ then it is continuous at $x=a$.

How can a function fail to be differentiable?

1. If not continuous, or,
2. If $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{DNE or } \pm \infty$
Higher derivatives: \[ f''(x) = \frac{d^2f}{dx^2}, \quad f'''(x) = \frac{d^3f}{dx^3}, \quad f^{(n)}(x) = \frac{d^nf}{dx^n} \]

Alternate notation: \[ \frac{d^2f}{dx^2} = \frac{d}{dx} \left( \frac{df}{dx} \right) \] helps with units.

e.g. if \( f \) has units meters & \( x \) units of seconds,
then \[ \frac{df}{dx} = f'(x) \text{ has units m/s, (speed)} \]
\[ \frac{d^2f}{dx^2} = f''(x) \text{ has units m/s}^2 \text{ (acceleration)} \]
\[ \frac{d^3f}{dx^3} = f'''(x) \text{ has units m/s}^3 \text{ "metersper second cubed"} \]

* Compute \( f'(x) \) and \( f''(x) \) for \( f(x) = |x| x = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases} \)

Should find: \( f'(x) = 2|x| = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -2x & \text{if } x < 0 \end{cases} \)

and \( f''(x) = \begin{cases} 2 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases} \)

Check: \[ \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \text{DNE} \text{, so } f''(0) \text{ does not exist!} \]