§3.1 Power law if \( n \) is constant, then \( \frac{d}{dn} x^n = n x^{n-1} \).

Linearity if \( a, b \) constant, then \( \frac{d}{dn} (a f(x) + b g(x)) = a f'(x) + b g'(x) \).

\( \Rightarrow \) Can differentiate any polynomial.

Exponentials \( \lim_{h \to 0} \frac{e^{h} - 1}{h} = 1 \Rightarrow \frac{d}{dn} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \).

§3.2 Product Rule \( \frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \)

or \( (uv)' = u'v + uv' \).

Quotient Rule \( \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \)

or \( \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2} \).

§3.3 In radians, arc length of a circle radius 1 equals angle.

\[ \frac{1}{\sinh h} \]

\( \cosh < \sinh < h \) (height small \( \Delta < \text{arc} \))

and \( \frac{h}{2\pi} \leq \frac{1}{2} \tanh \) (area segment < area \( \Delta \))

Rearrange: \( \cosh < \frac{\sinh}{h} < 1 \)

\( h \to 0 \)

\( \frac{h}{1} \)

So \( \lim_{h \to 0} \frac{\sinh}{h} = 1 \) by Squeeze theorem.

Use to compute \( \frac{d}{dn} \sin x = \cos x \).
\[ \lim_{n \to 0} \frac{\sin^3 n}{3n} = 1 \text{ etc.} \quad \lim_{n \to 0} \frac{\sin 2n}{3n} = \lim_{n \to 0} \frac{\sin 2n - 4n + 2n}{2n \sin 4n} \cdot \lim_{n \to 0} \frac{2n}{\sin 4n} \cdot \lim_{n \to 0} \frac{2n}{n} = 1 \cdot 1 \cdot \frac{2}{4} = \frac{1}{2}. \]

Other trig functions:
\[ \frac{d}{dx} \cos x = -\sin x. \]
\[ \frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \sec^2 x \]
\[ \frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \sec x \tan x \]
\[ \frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = -\csc x \cot x \]
\[ \frac{d}{dx} \cot x = \frac{d}{dx} \frac{1}{\tan x} = -\frac{d}{dx} \cos x = -\csc^2 x \]

\[ \frac{d}{dx} \] 3.4 Chain rule: \[ \frac{d}{dx} f(g(x)) = \frac{d}{dx} f(g(x)) \cdot g'(x) \]

or \[ \frac{d}{dx} \left( f(g(x)) \right) = \frac{df}{dx} \cdot \frac{dg}{dx}. \]

Derivatives of: \( \sin x, \cos x, \text{ etc.} \)
\( e^x, \alpha^x = e^x \ln \alpha \)

3.5 Implicit diff: \[ \frac{d}{dx} f(y) = f'(y) \cdot \frac{dy}{dx}, \]

E.g. \[ \frac{d}{dx} (xy^2) = 1 \cdot y^2 + x \cdot \frac{d}{dx} y^2 \quad (\text{Product Rule}) \]
\[ = y^2 + x \cdot 2y \cdot \frac{dy}{dx} \quad (\text{Implicit diff/Chain rule}) \]

Use to find slope \( \frac{dy}{dx} \) of implicit curves, e.g. \( xy^2 = 3 \cos(xy) \).

\* Diff both sides with respect to \( x \), and solve for \( \frac{dy}{dx} \).
Inverse functions: \[ y = f^{-1}(x) \iff f(y) = x \] \[ \Rightarrow \frac{d}{dx} f(y) = \frac{d}{dx} x \] \[ \Rightarrow f'(y) \frac{dy}{dx} = 1 \] \[ \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))} \]

Apply to \( y = \sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \) etc.

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{-1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}, \text{etc...} \]

§3.6 \[ \frac{d}{dn} \ln u = \frac{1}{u} = u^{-1} \] (Inverse function of \( e^u \) ....)

Generally: \[ \frac{d}{dn} \log_a x = \frac{d}{dn} \frac{\ln x}{\ln a} = \frac{1}{u \ln a} \]

Logarithmic differentiation: \[ \frac{d}{dn} (\ln f(x)) = \frac{f'(x)}{f(x)} \Rightarrow f'(x) = f(x) \frac{d}{dn} (\ln f(x)) \]

Useful when \( \ln f(x) \) is simpler than \( f(x) \).

\( \text{Ex:} \) If \( f(x) = e^{\tan x \cdot x^3} \), then \( \ln f(x) = \tan x \cdot x^3 \),

so \[ \frac{d}{dn} \ln f(x) = \frac{d}{dn} (\tan x \cdot x^3) \]

\[ \Rightarrow \frac{d}{dn} f(x) = e^{\tan x \cdot x^3} \left[ \tan x \cdot 3x^2 + x^3 \sec^2 x \right] \]

Apply to: \( f(x) = x^x, x^\sin x, \frac{\sqrt{3x^2 + 1}}{(4x+1)^3 (7x-1)^5}, \text{etc...} \)
\[ e \text{ as a limit: } e^n = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \]

or \[ e = \lim_{n \to 0} (1 + x)^{\frac{1}{n}}. \]

3.8 Natural growth \((k > 0)\) / decay \((k < 0)\) equation:

\[
\frac{dy}{dx} = ky. \quad \text{Solution: } y(x) = y(0) \cdot e^{kx}.
\]

Rate of change of quantity \(y\) proportional to size of \(y\).

E.g. Population growth, rate of spread of disease, half-life/radiation, continuously compounded interest.

**Cooling (Newton)**

\[
\frac{dT}{dt} = -k(T - T_s), \quad k > 0 \text{ constant } \quad T_s = \text{surrounding temp}.
\]

Rate of change of temp proportional to difference between temp and that of surroundings...

Solution: \[ T(t) = T_s + (T(0) - T_s)e^{-kt} \]

\[ T(t) \uparrow \]

\[ T(0) \]

\[ T_s \]
3.9. Related Rates: Word problems for Chain Rule or Implicit Differentiation.

- Draw picture.
- Choose variables.
- List everything in the question in terms of your variables.
- Differentiate.
- Solve for the desired quantity and interpret solution in context of the question.

3.10. Tangent line equation

\[ y = f(a) + f'(a) (x-a) \]

For \( x \) close to \( a \), \( f(x) \approx f(a) + f'(a) (x-a) \).

Function approximated by its tangent line.

Notation: \( L_a(x) = f(a) + f'(a) (x-a) \).
Linear approximation at \( a \).

Use: Approximate roots, e.g. \( \sqrt{4.2} = \sqrt{4 + \frac{1}{5}} \),

so \( f(4 + \frac{1}{5}) = f(4) + f'(4) (4 \cdot \frac{1}{5} - 4) \)

\[ = f(4) + \frac{1}{5} f'(4), \]

where \( f(x) = \sqrt{x} \).

Differentials: \( y = f(x) \Rightarrow dy = f'(x) dx \)
\[ dy = f'(x) \, dx. \]

\[ \uparrow \quad \uparrow \]

\[ \text{small change in } x \quad \text{leads to...} \]

\[ \text{small change in } y \]

Eg. If \[ y = 17x^2 + 3x - 1 \], and \[ dx = \frac{1}{2} \] at \[ x = 1 \],
then the resulting small change in \( y \) is approximately
\[ dy = (34x + 3) \, dx \]
\[ = (34 \cdot 1 + 3) \frac{1}{2} = \frac{37}{2}. \]