§3.1 Power law: if n is constant, then $\frac{d}{dx}x^n = nx^{n-1}$.

Linearity: if a,b constant, then $\frac{d}{dx}\left(af(x)+bg(x)\right) = af'(x)+bg'(x)$.

⇝ Can differentiate any polynomial.

Exponentials: $\lim_{h\to 0}\frac{e^h-1}{h} = 1 \Rightarrow \frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h}$

$$= e^x.$$

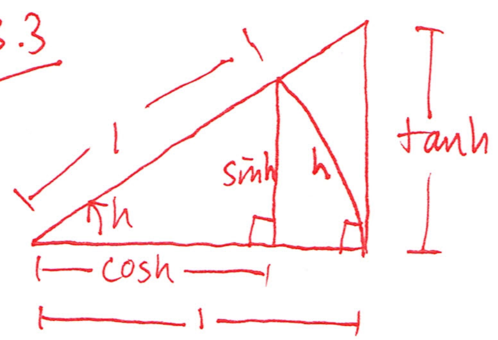
§3.2 Product Rule: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

or $(uv)' = u'v+uv'$.

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

or $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$.

§3.3



In radians, arclength of a circle radius 1 equals angle.

$\sin h < h$ (height small $\triangle$ < arc)

and $\frac{h}{2\pi}\cdot\pi < \frac{1}{2}\tan h$ (area segment < area big $\triangle$)

Rearrange: $\cos h < \frac{\sin h}{h} < 1$

$$\begin{array}{ccc} \cos h & & 1 \\ \downarrow{\scriptstyle h\to 0} & & \downarrow{\scriptstyle h\to 0} \\ 1 & & 1 \end{array}$$

so $\lim_{h\to 0}\frac{\sin h}{h} = 1$ by squeeze theorem.

Use to compute $\frac{d}{dx}\sin x = \cos x$.

Limits: $\lim_{x\to 0} \frac{\sin 3x}{3x} = 1$ etc. $\lim_{x\to 0} \frac{\sin 2x}{\sin 4x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{4x}{\sin 4x} \cdot \frac{2x}{4x}$

$= 1 \cdot 1 \cdot \frac{2}{4} = \frac{1}{2}$.

Other trig functions: $\frac{d}{dx} \cos x = -\sin x$.

$$\left.\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} = \sec^2 x \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} = \sec x \tan x \\
\frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{\sin x} = -\csc x \cot x \\
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{1}{\tan x} = \frac{d}{dx}\frac{\cos x}{\sin x} = -\csc^2 x
\end{aligned}\right\}$$ Use quotient rule to justify.

§3.4 Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

or $\frac{d}{dx}((f\circ g)(x)) = \frac{df}{dg} \cdot \frac{dg}{dx}$.

Derivatives of: $\sin kx, \cos kx$, etc,

e^{kx}, $a^x = e^{x \ln a}$.

§3.5 Implicit diff: $\frac{d}{dx} f(y) = f'(y) \cdot \frac{dy}{dx}$,

e.g. $\frac{d}{dx}(xy^2) = 1 \cdot y^2 + x \cdot \frac{d}{dx} y^2$ (Product Rule)

$= y^2 + x \cdot 2y \cdot \frac{dy}{dx}$ (Implicit diff/Chain rule)

Use to find slope $\frac{dy}{dx}$ of implicit curves, e.g. $x^2y^3 = 3\cos(xy)$.

- Diff both sides with respect to x, and solve for $\frac{dy}{dx}$.

Inverse functions: $y = f^{-1}(x) \iff f(y) = x$

$$\Rightarrow \frac{d}{dx} f(y) = \frac{d}{dx} x$$

$$\Rightarrow f'(y) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Apply to $y = \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, etc.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{-1}{\sqrt{1-x^2}}, \frac{1}{1+x^2}, \text{ etc...}$$

§3.6 $\frac{d}{dx} \ln x = \frac{1}{x} = x^{-1}$ (Inverse function of e^x....)

Generally: $\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{x(\ln a)}$.

Logarithmic differentiation:

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)} \Longrightarrow f'(x) = f(x) \frac{d}{dx}(\ln f(x)).$$

Useful when $\ln f(x)$ is simpler than $f(x)$.

Ex If $f(x) = e^{7\tan x \cdot x^3}$, then $\ln f(x) = \underbrace{7\tan x \cdot x^3}_{\text{easy to diff...}}$,

so $\frac{d}{dx} \ln f(x) = 7\sec^2 x \cdot x^3 + 7\tan x \cdot 3x^2$

$$\Rightarrow f'(x) = e^{7\tan x \cdot x^3}\left[7x^3 \sec^2 x + 21x^2 \tan x\right].$$

Apply to: $f(x) = x^x, x^{\sin x}, \frac{\sqrt{3x^2+2}}{(4x+1)^{1/3}(7x-1)^{1/5}}$, etc...

e as a limit: $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

or $e = \lim_{x\to 0}(1+x)^{1/x}$.

§3.8 Natural growth ($k>0$) / decay ($k<0$) equation:

$\frac{dy}{dx} = ky$. Solution $y(x) = y(0)\cdot e^{kx}$.

rate of change of quantity y proportional to size of y.

Eg. Population growth, rate of spread of disease, half-life/radiation, continuously compounded interest.

Cooling (Newton)

$$\frac{dT}{dt} = \underbrace{-k(T-T_s)}, \qquad k>0 \text{ constant}, \quad T_s = \text{surrounding temp.}, \quad T(t) = \text{temp at time } t.$$

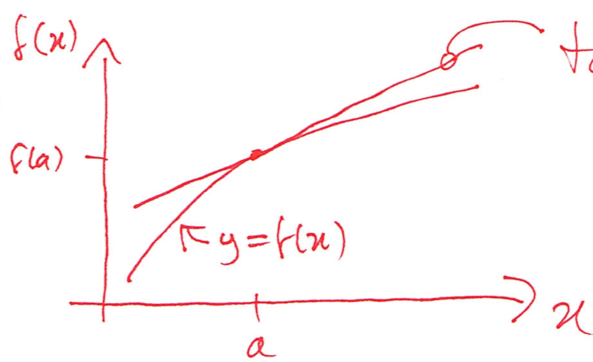
rate of change of temp proportional to <u>difference</u> between temp and that of surroundings....

Solution: $T(t) = T_s + (T(0)-T_s)e^{-kt}$

§3.9 Related Rates: Word problems for Chain Rule or Implicit Differentiation.

- Draw picture.
- Choose variables
- List everything in the question in terms of your variables.
- Differentiate.
- Solve for the desired quantity and interpret solution in context of the question.

§3.10



tangent line equation
$$y = f(a) + f'(a)(x-a)$$

For x close to a, $f(x) \approx f(a) + f'(a)(x-a)$.

Function approximated by its tangent line.

Notation: $L_a(x) = f(a) + f'(a)(x-a)$.

linear approximation at a.

Use: Approximate roots, ~~etc~~ e.g. $\sqrt{4.2} = \sqrt{4 + \frac{1}{5}}$,

so $f(4+\frac{1}{5}) = f(4) + f'(4)(\cancel{4}\frac{1}{5} - 4)$

$= f(4) + \frac{1}{5} f'(4)$,

where $f(x) = \sqrt{x}$.

Differentials: $y = f(x) \Rightarrow dy = f'(x)\,dx$

$$dy = f'(x)\,dx.$$

small change in x leads to...

... small change in y

E.g. If $y = 17x^2 + 3x - 1$, and $dx = \frac{1}{2}$ at $x = 1$, then the resuling small change in y is approximately

$$dy = (34x+3)\,dx$$

$$= (34\cdot 1 + 3)\frac{1}{2} = \frac{37}{2}.$$