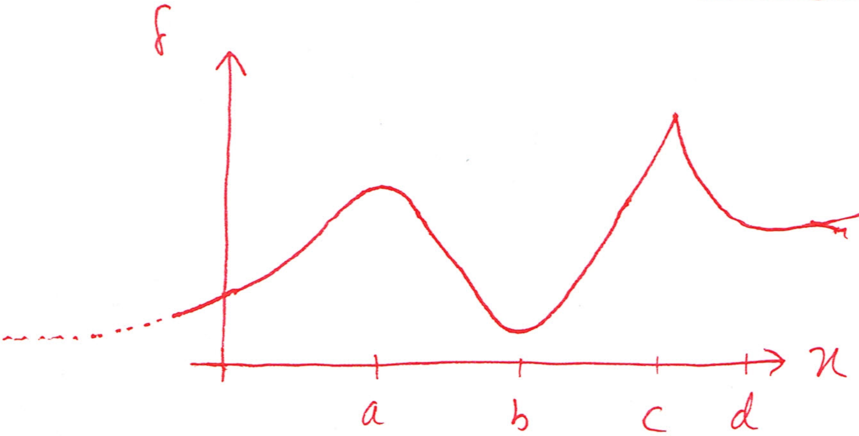
§4.1 Def^n $f(c)$ is the absolute maximum of f if $f(c) \geq f(x)$ for all x.

" " " " minimum of f " $f(c) \leq f(x)$ " " "

" " a local maximum of f if $f(c) \geq f(x)$ for all x near c

" " " " minimum of f if $f(c) \leq f(x)$ for all x near c.



abs max : $f(c)$

abs min : $f(b)$

local max : $f(a)$, $f(c)$

local min : $f(b)$, $f(d)$

Extreme Value theorem If f is continuous on $[a,b]$ then f is bounded, and there exist $m, M \in [a,b]$ such that

$f(m)$ is the absolute minimum, and

$f(M)$ is the absolute maximum.

Def^n A critical value c of a function is a value for which $f'(c) = 0$ or $f'(c)$ does not exist.

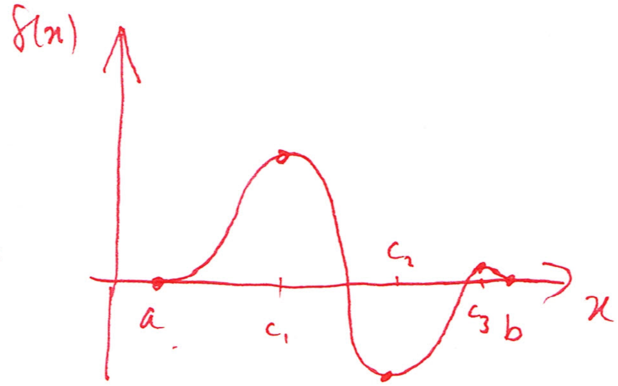
Fact: All local max/min are at critical values.

Converse ~~fals~~ false: e.g. $f(x) = x^3$ has critical value $x = 0$,

but $f(0)$ not a local max or min

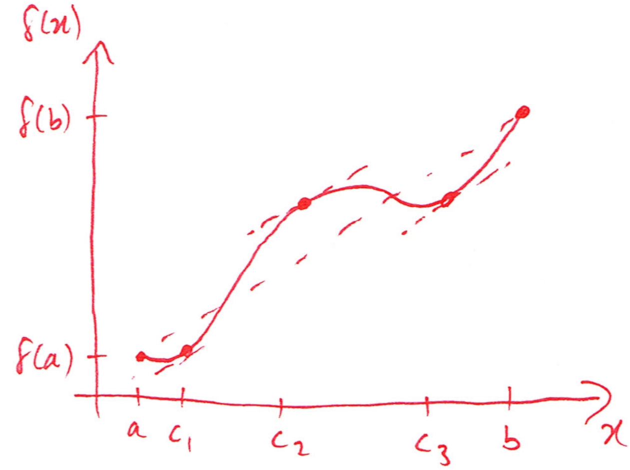
Thm If f is continuous on a closed bounded interval $[a,b]$, then the absolute maximum and minimum are located either at the endpoints of $[a,b]$ or at critical values.

§4.2 **Rolle's Thm** Suppose f is continuous on $[a,b]$, differentiable on (a,b), and $f(a)=f(b)=0$. Then there exists at least one $c \in (a,b)$ such that $f'(c)=0$



Mean Value Thm (MVT)

Suppose f continuous on $[a,b]$ and differentiable on (a,b) then there exists at least one $c \in (a,b)$ such that



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Slope at c — average slope of f.

Existence of roots: Between every pair of roots of $f(x)=0$, there is a solution to $f'(x)=0$ (look at Rolle's thm pic)

Use: If $f'(x)=0$ has three solutions, then $f(x)=0$ has at most four solutions (but might have fewer)

Functions with same derivative: If $f'(x)=g'(x)$ on an interval I, then $g(x)=f(x)+C$ for some constant C.

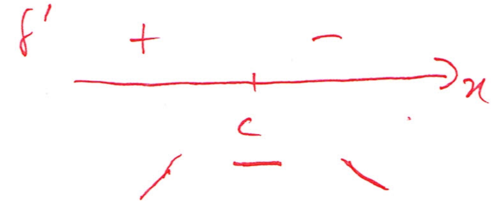
E.g. All functions whose derivative is $f'(x)=x^2$ on $\mathbb{R}$ are $f(x)=\frac{1}{3}x^3+C$.

§4.3 Thm $f' > 0 \Rightarrow f$ increasing

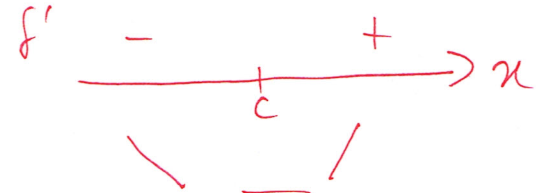
$f' < 0 \Rightarrow f$ decreasing

1st derivative test Suppose f continuous on $[a,b]$, and that $c \in (a,b)$ is a critical value. If f' changes sign at $x=c$, then:

- f' changes pos $\rightarrow$ neg
  $f(c)$ a local maximum



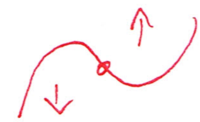
- f' chages neg $\rightarrow$ pos
  $f(c)$ a local minimum



Def^{n} f is concave up if $f'' > 0$

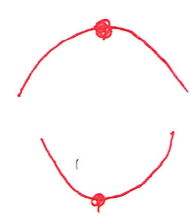
f is concave down if $f'' < 0$

c is an inflection point if f'' changes sign at $x=c$.



Second derivative test Let $f'(c)=0$ where f is continuous on (a,b), and $c \in (a,b)$. Then

- $f''(c) < 0 \Rightarrow f(c)$ a local maximum
- $f''(c) > 0 \Rightarrow f(c)$ a local minimum



If $f''(c)=0$, then no conclusion. e.g. $f(x) = x^3, x^4, -x^4$ all have $f'(0) = 0 = f''(0)$. x^4 has local minimum at $x=0$

$-x^4$ has local maximum at $x=0$

x^3 has neither

§4.4 An <u>indeterminate form</u> is a limit of the form:

(limits can be 1-sided, or at $\pm\infty$)

<u>Type</u>	<u>Limit</u>	<u>Condition</u>
$\frac{0}{0}$	$\lim_{x\to a} \frac{f(x)}{g(x)}$	$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$
$\frac{\infty}{\infty}$	$\lim_{x\to a} \frac{f(x)}{g(x)}$	$\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$
$0\cdot\infty$	$\lim_{x\to a} f(x)g(x)$	$\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=\infty$.
$\infty-\infty$	$\lim_{x\to a} f(x)-g(x)$	$\lim_{x\to a} f(x)=\infty=\lim_{x\to a} g(x)$
0^0	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$
∞^0	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=0$
1^∞	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x)=1$, $\lim_{x\to a} g(x)=\infty$.

<u>Thm</u> (L'Hôpital's Rule).

Suppose that $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and suppose that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

- <u>Not</u> quotient rule!
- If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \text{DNE}$ then rule doesn't apply, e.g. $\lim_{x\to\infty} \frac{x+\cos x}{x} = 1$.
- <u>Think</u> before applying: can you use an elementary method?
- <u>Only</u> applies to indeterminate forms, e.g. $\lim_{x\to 0^+} \frac{2x-1}{x} \neq \lim_{x\to 0^+} \frac{2}{1} = 2$

Type $0\cdot\infty$ $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)}$ $\leftarrow$ type $\frac{0}{0}$

or $= \lim_{x\to a} \frac{g(x)}{1/f(x)}$ $\leftarrow$ type $\frac{\infty}{\infty}$.

Type $\infty-\infty$ Let $F(x) = \frac{1}{f(x)}$, $G(x) = \frac{1}{g(x)}$, then

$$\lim_{x\to a} f(x)-g(x) = \lim_{x\to a} \frac{1}{F(x)} - \frac{1}{G(x)} = \lim_{x\to a} \frac{G(x)-F(x)}{F(x)G(x)} \leftarrow \text{type } \frac{0}{0}.$$

Type 0^0 Take logs: $\ln\left(\lim_{x\to a} f(x)^{g(x)}\right) = \lim_{x\to a} \ln\left(f(x)^{g(x)}\right)$ (since $\ln$ continuous)

$$= \lim_{x\to a} g(x)\cdot \ln f(x) \leftarrow \text{type } 0\cdot\infty$$

Type ∞^0 $\ln\left(\lim_{x\to a} f(x)^{g(x)}\right) = \lim_{x\to a} g(x)\cdot\ln f(x)$ $\leftarrow$ type $0\cdot\infty$

Type 1^∞ $\ln\left(\lim_{x\to a} f(x)^{g(x)}\right) = \lim_{x\to a} g(x)\cdot\ln f(x)$ $\leftarrow$ type $\infty\cdot 0$.

Remember to take $e^{\lim_{x\to a} g(x)\cdot \ln f(x)}$ for your final answer. Similar to logarithmic differentiation.

§4.5 When sketching a curve, consider all the following, if they apply.

1. Domain of f : what values of x are allowed?
2. x,y-intercepts : where does curve cross x,y-axes?
3. Asymptotes : Horizontal $\lim_{x \to \pm\infty} f(x) = ?$

   Vertical ; any a with $\lim_{x \to a^\pm} f(x) = \pm\infty$?

   Slant : $\lim_{x \to \pm\infty} f(x) \sim ax+b$?
4. Critical values : $f'(c) = 0$ or DNE.
5. Local max/min : 1st or 2nd derivative tests.
6. Intervals of $\underbrace{\text{increase/decrease}}_{\text{sign of } f'}/\underbrace{\text{concavity}}_{\text{sign of } f''}$.
7. Periodicity : e.g. $f(x) = \sin x$.
8. Is f even or odd

   even: $f(-x) = f(x)$ odd: $f(-x) = -f(-x)$.

§4.7 Optimization: Find maxima / minima in word problems.

- Draw a picture } if necessary.
- Assign variables }
- Restate problem "Find maximum (or minimum) of $f(x) = ?$ for $a \leq x \leq b$"
- Interpret answer in language/units/context of original problem.

§4.9 Def^n $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Thm (MVT) If F, G are anti-derivatives of f on the same interval, then $G(x) = F(x) + C$ for some constant C.

"Guess and differentiate, then add C"

Don't memorize lists of anti-derivatives!

e.g.

$f(x)$	anti-derivatives
x^2	$\frac{1}{3}x^3 + C$
$\sin 2x$	$-\frac{1}{2}\cos 2x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$2x\cos(x^2)$	$\sin(x^2) + C$

Kinematics

If $h''(t) = -g$ (gravity constant)

$h'(0) = v_0$ (initial velocity)

$h'(0) = y_0$ (initial height)

Then,

$v(t) = h'(t) = v_0 - gt$

and,

$h(t) = y_0 + v_0 t - \frac{1}{2}gt^2$.