1 Functions and Models

1.1 Four Ways to Represent a Function

1. Temperature readings $T$ (in °F) were recorded every two hours from midnight to 2pm. Time $t$ was measured in hours from midnight.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>82</td>
<td>75</td>
<td>74</td>
<td>75</td>
<td>84</td>
<td>90</td>
<td>93</td>
<td>94</td>
</tr>
</tbody>
</table>

(a) Use the readings to sketch a rough graph of $T$ as a function of $t$.

(b) Use your graph to estimate the temperature at 9pm.

2. A spherical balloon with radius $r$ inches has volume $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate the balloon from a radius of $r$ inches to a radius of $r + 1$ inches.

3. Find the domain of the function $f(x) = \frac{x^4 + 4}{x^2 - 9}$.

4. Find the domain of the function $f(x) = \frac{2x^5 - 5}{x^3 + x - 6}$.

5. Find the domain of the function $f(u) = \frac{u + 1}{1 + \pi u}$.

6. Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2, \\ 2x - 5 & \text{if } x > 2. \end{cases}$$

7. Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x + 9 & \text{if } x < -3, \\ -2x & \text{if } |x| \leq 3, \\ -6 & \text{if } x > 3. \end{cases}$$

8. A box (without lid) is to be made by cutting squares of side-length $x$ in from the corners of a piece of card which is 12in by 20in and folding up the edges. Find the volume $V$ of the box as a function of $x$.

9. A cell phone plan has a basic charge of $35 per month and includes 400 free minutes and charges 10 cents per additional minute. Find and graph the monthly cost of the plan $C$ as a function of the number of used minutes $x$ for $0 \leq x \leq 600$. 
1.2 Mathematical Models: A Catalog of Essential Functions

1. What do all members of the family of linear functions \( f(x) = 1 + mx + 3 \) have in common? Sketch several members of the family.

2. The average surface temperature of the earth is modeled by \( T = 0.02t + 8.50 \) where \( T \) is the temperature in \(^\circ\)C and \( t \) represents years since 1900.
   (a) What do the slope and \( T \)-intercept represent?
   (b) Use the equation to predict the average global surface temperature in 2100.

3. The relationship between the Fahrenheit (\( F \)) and Celcius (\( C \)) temperature scales is given by the linear function \( F = \frac{9}{5}C + 32 \).
   (a) Sketch a graph of this function.
   (b) What is the slope of the graph and what does it represent? What is the \( F \)-intercept and what does it represent?

4. Many physical quantities are connected by inverse square laws, that is, by power functions of the form \( f(x) = kx^{-2} \), where \( k \) is constant. I.e. the illumination of an object by a light source is inversely poroportional to the square of the distance from the source. Imagine after dark you are reading a book illuminated by a single light which is too dim. You move halfway towards the light. How much brighter is the lamp?

5. Ecologists have modeled the species-of-bat-per-unit-area relationship with a power function \( S = 0.7A^{0.3} \), where \( S \) is the number of species living in an area \( A \).
   (a) If a cave has area 60m\(^2\), how many species would you expect to find in the cave?
   (b) If only four species of bat live in a cave, estimate the area of the cave.

1.3 New Functions from Old Functions

1. Graph the function \( y = (x - 1)^3 \) by transforming the graph of a standard function.

2. Graph the function \( y = 4 \sin 3x \) by transforming the graph of a standard function.

3. Graph the function \( y = 1 - 2\sqrt{x + 3} \) by transforming the graph of a standard function.

4. A variable star has time between periods of maximum brightness of 5.4 days, average brightness 4.0 and the brightness varies by \( \pm 0.35 \) magnitude. Find a function which models the brightness as a function of time.

5. Find the functions \( f \circ g \), \( g \circ f \), \( f \circ f \) and \( g \circ g \) and their domains for the following pairs of functions: \( f(x) = 1 - 3x \), \( g(x) = \cos x \).

6. Find the functions \( f \circ g \), \( g \circ f \), \( f \circ f \) and \( g \circ g \) and their domains for the following pairs of functions: \( f(x) = \sqrt{x} \), \( g(x) = \sqrt{1 - x} \).

7. A spherical balloon is being inflated and the radius is increasing at a rate of 2cm/s.
   (a) Express the radius \( r \) of the balloon as a function of the time \( t \) in seconds.
   (b) If \( V \) is the volume of the balloon as a function of the radius, find \( V \circ r \) and interpret it.
1.4/5 Exponential Functions

1. Use the law of Exponents to rewrite and simplify the expressions:
   
   (a) \(8^{4/3}\), \hspace{1cm} (b) \(x(3x^2)^3\).

2. Starting with the graph of \(y = e^x\), find the equation of the graph that results from
   
   (a) Reflecting about the line \(y = 4\).
   (b) Reflecting about the line \(x = 2\).

3. Find the domain of each function:
   
   (a) \(g(t) = \sin(e^{-t})\), \hspace{1cm} (b) \(g(t) = \sqrt{1 - 2^t}\).

1.5/6 Inverse Functions and Logarithms

1. Is the function \(f(x) = 10 - 3x\) 1–1? What about \(g(x) = \cos x\)? Justify your answers.

2. If \(f(x) = x^5 + x^3 + x\), find \(f^{-1}(3)\) and \(f(f^{-1}(2))\).

3. Find a formula for the inverse of the function \(f(x) = \frac{4x-1}{2x^3}\).

4. Find the exact values of the expressions
   
   (a) \(e^{-2\ln 5}\), \hspace{1cm} (b) \(\ln(\ln(e^{10}))\).

5. When a camera flash goes off, the batteries immediately begin to recharge the flash’s capacitor, which stores charge given by
   
   \[Q(t) = Q_0(1 - e^{-t/a})\].

   (The maximum charge capacity is \(Q_0\) and \(t\) is measured in seconds.)
   
   (a) Find the inverse of this function and explain its meaning.
   (b) How long does it take to recharge the capacitor to 90% of capacity if \(a = 2\)?

6. Simplify the expression \(\cos(2 \tan^{-1} x)\).