Math 2A Single Variable Calculus Homework Questions 2

2 Limits and Derivatives

2.1 The Tangent and Velocity Problems

1. If a ball is thrown into the air with a velocity of 40ft/s, its height in feet t seconds later is given by $y = 40t - 16t^2$.

   (a) Find the average velocity for the time period beginning when $t = 2$ and lasting
       (i) 0.5s,   (ii) 0.1s,   (iii) 0.05s,   (iv) 0.01s.
   (b) Estimate the instantaneous velocity when $t = 2$.

2. The displacement (in cm) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2\sin(\pi t) + 3\cos(\pi t)$, where t is measured in seconds.

   (a) Find the average velocity for the following time periods
       (i) $[1, 2]$,   (ii) $[1, 1.1]$,   (iii) $[1, 1.01]$,   (iv) $[1, 1.001]$.
   (b) Estimate the instantaneous velocity when $t = 1$.

2.2 The Limit of a Function

1. Explain in your own words what is meant by the equation

   $$\lim_{x \to 2} f(x) = 5.$$

   Is it possible for this to be true and yet $f(2) = 3$?

2. Guess the value of the limit $\lim f$ (if it exists) by evaluating the function f at the given numbers (to 6dp).

   $$\lim_{x \to -1} \frac{x^2 - 2x}{x^2 - x - 2}, \qquad x = 0, -0.5, -0.9, -0.95, -0.99, -.0999, -2, -1.5, -1.1, -1.0, -1.001.$$

3. Guess the value of the limit $\lim f$ (if it exists) by evaluating the function f at the given numbers (to 6dp).

   $$\lim_{h \to 0} \frac{(2 + h)^5 - 32}{h}, \qquad h = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001.$$

4. Determine the infinite limit $\lim\limits_{x \to -3^-} \frac{x+2}{x+3}$.

5. Determine the infinite limit $\lim\limits_{x \to 5^-} \frac{e^x}{x-5}$.

6. Determine the infinite limit $\lim\limits_{x \to 2^-} \frac{x^2 - 2x}{x^2 - 4x + 4}$.

7. In the theory of relativity, the mass of a particle with velocity v is

   $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

   where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

2.3 Calculating Limits using the Limit Laws

1. Given that

$$\lim_{x \to 2} f(x) = 4, \qquad \lim_{x \to 2} g(x) = -2, \qquad \lim_{x \to 2} h(x) = 0,$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2}[f(x) + 5g(x)]$.

(b) $\lim_{x \to 2}[g(x)]^3$.

(c) $\lim_{x \to 2} \sqrt{f(x)}$.

(d) $\lim_{x \to 2} \frac{3f(x)}{g(x)}$.

(e) $\lim_{x \to 2} \frac{g(x)}{h(x)}$.

(f) $\lim_{x \to 2} \frac{g(x)h(x)}{f(x)}$.

2. Evaluate $\lim_{x \to -1}(x^4 - 3x)(x^2 + 5x + 3)$ and justify each step by indicating the appropriate Limit Law(s).

3. Evaluate $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$ and justify each step by indicating the approriate Limit Law(s).

4. Evaluate $\lim_{t \to 2} \left(\frac{t^2 - 2}{t^3 - 3t + 5} \right)^2$ and justify each step by indicating the approriate Limit Law(s).

5. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3.$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2}(x + 3)$$

is correct.

6. Evaluate the limit, if it exists: $\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$.

7. Evaluate the limit, if it exists: $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$.

8. Evaluate the limit, if it exists: $\lim_{h \to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$.

9. If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

10. Prove that $\lim_{x \to 0^+} \sqrt{x}[1 + \sin^2(2\pi/x)] = 0$.

11. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

12. Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

13. Is there a number a such that

$$\lim\limits_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

2.5 Continuity

1. Explain why each function is continuous or discontinuous.

   (a) The temperature at a specific location as a function of time.

   (b) The temperature at a specific time as a function of the distance due east of Disneyland.

   (c) The altitude above sea level as a function of the distance due east of Disneyland.

   (d) The cost of a taxi ride as a function of the distance traveled.

   (e) The current in the circuit for the lights in a room as a function of time.

2. Suppose f, g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2}[3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

3. Use the definition of continuity and the properties of limits to show that the function $g(x) = 2\sqrt{3-x}$ is continuous on the interval $(-\infty, 3]$.

4. Explain why $f(x) = \frac{1}{x+2}$ is discontinuous at $a = -2$. Sketch the graph of f.

5. Explain why the following function f is discontinuous at $a = 3$.

$$f(x) = \begin{cases} \frac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3, \\ 6 & \text{if } x = 3. \end{cases}$$

   Sketch the graph of f.

6. How would you remove the discontinuity at $x = 2$ of the function $f(x) = \frac{x^3 - 8}{x^2 - 4}$?

7. Explain why the function $h(x) = \frac{\sin x}{x+1}$ is continuous at every number in its domain. State the domain.

8. Explain why the function $h(x) = \frac{e^{\sin x}}{2 + \cos \pi t}$ is continuous at every number in its domain. State the domain.

9. Use continuity to evaluate the limit $\lim_{x \to \pi} \sin(x + \sin x)$.

10. Show that the function f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi/4, \\ \cos x & \text{if } x \geq \pi/4. \end{cases}$$

11. For which values of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2, \\ x^3 - cx & \text{if } x \geq 2. \end{cases}$$

12. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(x) = 1000$.

13. Use the Intermediate Value Theorem to show that there is a root of the equation $\sin x = x^2 - x$ for $1 < x < 2$.

14. Use the Intermediate Value Theorem to show that there is a root of the equation $e^{x-1} = \frac{2}{x^2}$ for $1 < x < 2$.

15. (HARD!) For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational}, \\ x & \text{if } x \text{ is irrational}. \end{cases}$$

16. (HARD!) If a, b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

17. A Tibetan Monk leaves the monastery at 7am and takes his usual path to the top of the mountain, arriving at 7pm. The next morning he starts at the top of the mountain at 7am and takes the same path back, arriving at the monastery at 7pm. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2.6 Limits at Infinity: Horizontal Asymptotes

1. Evaluate $\lim_{x \to \infty} \dfrac{1 - x^2}{x^3 - x + 1}$ or show that it does not exist.

2. Evaluate $\lim_{t \to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$ or show that it does not exist.

3. Evaluate $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x} \right)$ or show that it does not exist.

4. Evaluate $\lim\limits_{x \to -\infty} \dfrac{1 + x^6}{x^4 + 1}$ or show that it does not exist.

5. Evaluate $\lim\limits_{x \to \infty} (x^2 - x^4)$ or show that it does not exist.

6. Evaluate $\lim\limits_{x \to \infty} \sqrt{x} \sin \dfrac{1}{x}$ or show that it does not exist.

7. Find the horizontal and vertical asymptotes of $y = \dfrac{1 + x^4}{x^2 - x^4}$.

8. Find the horizontal and vertical asymptotes of $F(x) = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$.

9. Find the horizontal and vertical asymptotes of $y = \dfrac{2e^x}{e^x - 5}$.

10. Find a formula for a function f that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

11. Find the horizontal asymptotes of the curve $y = \dfrac{x}{\sqrt{x^2+1}}$ and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

12. A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min.

    (a) Show that the concentration of salt after t minutes is
    $$C(t) = \frac{30t}{200 + t} \text{ g/L}.$$

    (b) What happens to the concentration as $t \to \infty$?

2.7 Derivatives and Rates of Change

1. Find an equation of the tangent line to the curve $y = x^3 - 3x + 1$ at $(2, 3)$.

2. Find an equation of the tangent line to the curve $y = \frac{2x+1}{x+2}$ at $(1, 1)$.

3. If a rock is thrown upwards on Mars with a velocity of 10m/s, its height in meters after t seconds is given by $H = 10t - 1.86t^2$.

    (a) Find the velocity of the rock after 1 second.
    (b) Find the velocity of the rock when $t = a$.
    (c) When will the rock hit the surface?
    (d) With what velocity will the rock hit the surface?

4. The displacement in meters of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

(a) Find the average velocity over each time interval:

$$(i)\ [3,4], \qquad (ii)\ [3.5,4], \qquad (iii)\ [4,5], \qquad (iv)\ [4,4.5].$$

(b) Find the velocity when $t = 4$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

5. If the tangent line to $y = f(x)$ at (4,3) passes through the point (0,2), find $f(4)$ and $f'(4)$.

6. If $g(x) = x^4 - 2$, find $g'(1)$ and thus the equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

7. Find $f'(a)$ if $f(t) = 2t^3 + t$.

8. Find $f'(a)$ if $f(x) = x^{-2}$.

9. Find $f'(a)$ if $f(x) = \frac{4}{\sqrt{1-x}}$.

10. The following limit represents the derivative of some function f at some number a. State a suitable f and a:

$$\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}.$$

11. The following limit represents the derivative of some function f at some number a. State a suitable f and a:

$$\lim_{t \to 1} \frac{t^4 + t - 2}{t - 1}.$$

12. A cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in exactly one hour. Torritcelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000 \left(1 - \frac{1}{60}t \right)^2.$$

Find the rate at which the water is flowing out of the tank as a function of t. What are its units? For times $t = 0, 10, 20, 30, 40, 50, 60$ mins, find the flow rate and the remaining water. Summarize your findings in a sentence or two. When is the flow rate the greatest? The least?

13. The quantity (in pounds) of coffee sold by a coffee company when the price is p dollars per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) Is $f'(8)$ positive or negative? Explain.

14. Determine whether $f'(0)$ exists if

$$f(x) \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

15. Determine whether $f'(0)$ exists if

$$f(x) \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2.8   The Derivative as a Function

1. If $f(x) = mx + b$, find the derivative using the definition of derivative and state the domain of the function and its derivative.

2. If $f(x) = 1.5x^2 - x + 3.7$, find the derivative using the definition of derivative and state the domain of the function and its derivative.

3. If $g(t) = \frac{1}{\sqrt{t}}$, find the derivative using the definition of derivative and state the domain of the function and its derivative.

4. If $f(x) = \frac{x^2-1}{2x-3}$, find the derivative using the definition of derivative and state the domain of the function and its derivative.

5. If $f(x) = x^{3/2}$, find the derivative using the definition of derivative and state the domain of the function and its derivative.

6. Let $P(t)$ be the percentage of Americans under the age of 18 at time t.The table gives the values for $(P(t)$ in census years from 1950 to 2000:

t	1950	1960	1970	1980	1990	2000
$P(t)$	31.1	35.7	34.0	28.0	25.7	25.7

   (a) What is the meaning of $P'(t)$ and what are its units?

   (b) Construct a table of estimated values for $P'(t)$.

   (c) Graph P and P' (approximately).

   (d) How would it be possible to get more accurate values for $P'(t)$?

7. Let $f(x) = \sqrt[3]{x}$.

   (a) If $a \neq 0$, use $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ to find $f'(a)$.

   (b) Show that $f'(0)$ does not exist.

   (c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0,0)$.

8. Show that the function $f(x) = |x - 6|$ is not differentiable at $x = 6$. Find a formula for f' and sketch its graph.

9. Let $f(x) = x|x|$.

(a) Sketch the graph of f.

(b) For what values of x is f differentiable?

(c) Find a formula for f'.

10. Recall that a function f is *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x. Prove the following:

(a) The derivative of an even function is an odd function.

(b) The derivative of an odd function is an even function.