Math 2A Single Variable Calculus Homework Questions 3

3.1 Derivatives of Polynomials and Exponential Functions

1. Differentiate $f(x) = \pi^2$.
2. Differentiate $h(x) = (x-2)(2x+3)$.
3. Differentiate $y = x^{5/3} - x^{2/3}$.
4. Differentiate $v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2$.
5. Differentiate $y = \sqrt[4]{t} - 4e^t$.
6. Differentiate $k(r) = e^r + r^e$.
7. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
   (a) Suppose that the pressure of a sample that occupies $0.106\,\mathrm{m}^3$ at $25°\mathrm{C}$ is $50\,\mathrm{kPa}$. Write V as a function of P.
   (b) Calculate $\frac{\mathrm{d}V}{\mathrm{d}P}$ when $P = 50\,\mathrm{kPa}$. WHat is the meaning of the derivative? What are its units?
8. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs:
   $$\text{(a)}\quad f(x) = x^n, \qquad \text{(b)}\quad f(x) = \frac{1}{x}.$$

3.2 The Product and Quotient Rules

1. Find the derivative of $f(x) = (1+2x^2)(x-x^2)$ in two ways: via the Product Rule and by multiplying out first. Do your answers agree?
2. Differentiate $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$.
3. Differentiate $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$.
4. Differentiate $y = \frac{t}{(t-1)^2}$.
5. Differentiate $y = A + \frac{B}{x} + \frac{C}{x^2}$.
6. Differentiate $f(x) = \frac{ax+b}{cx+d}$.
7. Find an equation for the tangent line to the curve $y = \frac{2x}{x+1}$ at (1,1).
8. If $g(x) = \frac{x}{e^x}$, find $g^{(n)}(x)$.
9. The curve $y = x/(1+x^2)$ is called a *serpentine.*
   (a) Find an equation of the tangent line at the point $(3, 0.3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line.

10. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (in yards) that is sold is a function of the selling price p (in dollars per yard), so we can write $q = f(p)$. The total revenue earned is then $R(p) = pf(p)$.

    (a) What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?

    (b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

11. At what numbers is the following function differentiable?

$$g(x) = \begin{cases} 2x & \text{if } x \leq 0, \\ 2x - x^2 & \text{if } 0 < x < 2, \\ 2 - x & \text{if } x \geq 2. \end{cases}$$

    Give a formula for g' and sketch the graphs of g and g'.

12. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

3.3 Derivatives of Trigonometric Functions

1. Differentiate $y = 2\sec x - \csc x$.

2. Differentiate $y = u(a\cos u + b\cot u)$.

3. Differentiate $y = \dfrac{\cos x}{1 - \sin x}$.

4. Prove that $\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x$.

5. Find an equation of the tangent line to the curve $y = (1 + x)\cos x$ at (0,1).

6. (a) Find an equation of the tangent line to the curve $y = 3x + 6\cos x$ at the point $(\pi/3, \pi + 3)$.

    (b) Illustrate part (a) by graphing the curve and the tangent line.

7. Find the points on the curve $y = \dfrac{\cos x}{2 + \sin x}$ at which the tangent is horizontal.

8. A mass on a spring vibrates horizontally on a smooth level surface with equation of motion $x(t) = 8\sin t$ where t is in seconds and x (in cm) is the distance of the mass to the right of its equilibrium point.

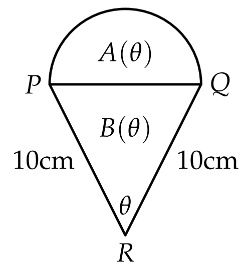
    (a) Find the velocity and acceleration at time t.

    (b) Find the position, velocity and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?

9. Find the derivative $\frac{\mathrm{d}^{35}}{\mathrm{d}x^{35}}(x\sin x)$ by calculating the first few derivatives and observing the pattern.

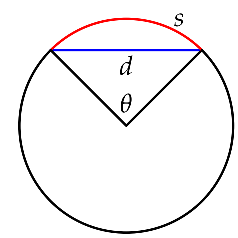
10. A semicircle with diameter PQ sits on an isosceles triangle PQR to form an ice-cream cone shape. If $A(\theta)$ is the area of the upper semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$$



11. (HARD!) The figure shows an arc of length s and a chord of length d both subtended by a central angle θ. Find

$$\lim_{\theta \to 0^+} \frac{s}{d}.$$



12. Let $f(x) = \dfrac{x}{\sqrt{1 - \cos 2x}}$.
    (a) Graph f. What type of discontinuity does it appear to have at 0?
    (b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

3.4 The Chain Rule

1. Let $y = \sin(\cot x)$. Write y in the form $f(g(x))$ and find the derivative $\frac{dy}{dx}$.
2. Find the derivative of $F(x) = (4x - x^2)^{100}$.
3. Find the derivative of $f(t) = \sqrt[3]{1 + \tan t}$.
4. Find the derivative of $F(t) = (3t - 1)^4 (2t + 1)^{-3}$.
5. Find the derivative of $y = \dfrac{\cos \pi x}{\sin \pi x + \cos \pi x}$.
6. Find the derivative of $F(v) = \left(\dfrac{v}{v^3 + 1} \right)^6$.
7. Find the derivative of $y = \sin(\sin(\sin x))$.
8. Find the first and second derivative of $y = \cos^2 x$.
9. The function $f(x) = \sin(x + \sin 2x)$, $0 \le x \le \pi$ arises in applications to frequency modulation synthesis. Calculate f' and graph it along with f using a computer or calculator.
10. Air is pumped into a spherical weather balloon. At time t the volume of the balloon is $V(t)$ and its radius is $r(t)$.
    (a) What do the derivatives $\frac{dV}{dr}$ and $\frac{dV}{dt}$ represent?
    (b) Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

11. Suppose $y = f(x)$ is a curve that always lies above the x-axis and never has a horizontal tangent and where f is differentiable everywhere. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x?

12. Use the chain rule to show that if θ is measured in degrees, then

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \sin\theta = \frac{\pi}{180} \cos\theta.$$

(This explains why we use *radians* in calculus and not degrees!)

3.5 Implicit Differentiation

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation if $2\sqrt{x} + \sqrt{y} = 3$.

2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation if $2x^3 + x^2y - xy^3 = 2$.

3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation if $\cos(xy) = 1 + \sin y$.

4. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation if $x \sin y + y \sin x = 1$.

5. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

6. Use implicit differentiation to find an equation of the tangent line to the hyperbola $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$.

7. Use implicit differentiation to find an equation of the tangent line to the asteroid $x^{2/3} + y^{2/3} = 4$ at the point $(-3\sqrt{3}, 1)$.

8. The curve with equation $y^2 = x^3 + 3x^2$ is called the *Tschirnhausen cubic.*

   (a) Find an equation for the tangent line to the curve at the point $(1, -2)$.
   (b) At what points does this curve have horizontal tangents?
   (c) Illustrate parts (a) and (b) in a single plot.

9. Show by implicit differentiation that the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y = 1.$$

10. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

11. The *van der Waals equation* for n moles of a gas is

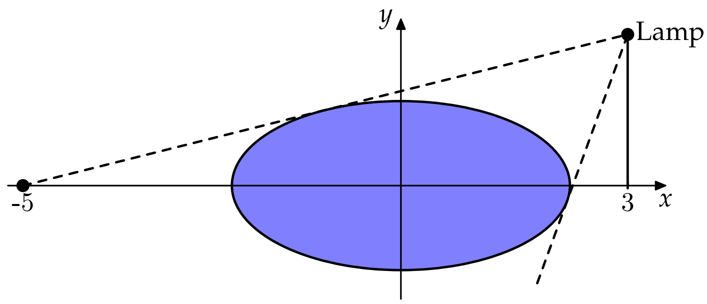
$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT,$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a, b are positive constants that are characteristics of a particular gas.

(a) If T remains constant, use implicit differentiation to find $\frac{dV}{dP}$.

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$L and a pressure of $P = 2.5$atm. Use $a = 3.592\text{L}^2\text{-atm/mole}^2$ and $b = 0.04267$L/mole.

12. A lamp is located at the point $(3, h)$, three units to the right of the y-axis and a shadow is created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



3.6 Derivatives of Logarithmic Functions

1–4 Differentiate the following functions:

1. $f(x) = \ln(\sin^2 x)$.
2. $f(u) = \dfrac{u}{1 + \ln u}$.
3. $g(r) = r^2 \ln(2r + 1)$.
4. $y = \log_2(e^{-x} \cos \pi x)$.
5. If $y = \dfrac{\ln x}{x^2}$, find y' and y''.
6. Find an equation for the tangent line to the curve $y = x^2 \ln x$ at the point $(1, 0)$.
7. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

8–11 Use logarithmic differentiation to compute the derivatives of the following functions:

8. $y = \dfrac{e^{-x} \cos^2 x}{x^2 + x + 1}$.
9. $y = \dfrac{(x+2)^{1/2}(x^2+1)^{1/3}}{(x^3+7)^{1/5}}$.
10. $y = x^{\cos x}$.
11. $y = (\ln x)^{\cos x}$.

3.8 Exponential Growth and Decay

1. A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.
   (a) What is the relative growth rate? Express your answer as a percentage.
   (b) What was the initial size of the culture?
   (c) Find an expression for the number of bacteria after t hours.
   (d) Find the number of cells after 4.5 hours.
   (e) Find the rate of growth after 4.5 hours.
   (f) When will the population reach 50,000?

2. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
   (a) What is the half-life of tritium-3?
   (b) How long would it take the sample to decay to 20% of its original amount?

3. A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

4. A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

3.9 Related Rates

1. A particle is moving along a hyperbola $xy = 8$. As it reaches the point (4,2), the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

3. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4 pm?

4. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising with the water is 6 inches deep?

5. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

6. If two resistors with resistances R_1 and R_2 are connected in parallel, then the total resistance R measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

   If R_1 and R_2 are increasing at rates of 0.3 Ω/s and 0.2 Ω/s respectively, how fast is R changing when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$?

7. Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (in cm) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm?

8. (HARD!) The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

3.10 Linear Approximations and Differentials

1. Find the linearization of $f(x) = \sin x$ at $x = \pi/6$.

2. Find the linearization of $f(x) = x^{3/4}$ at $x = 16$.

3. Find the differential of

$$\text{(a)} \quad y = \frac{s}{1+2s}, \qquad \text{(b)} \quad y = u \cos u.$$

4. Let $y = \cos \pi x$. Find the differential $\mathrm{d}y$ and evaluate it when $x = \frac{1}{3}$ and $\mathrm{d}x = -0.02$.

5. Let $y = \frac{x+1}{x-1}$. Find the differential $\mathrm{d}y$ and evaluate it when $x = 2$ and $\mathrm{d}x = 0.05$.

6. Use a linear approximation (or differentials) to estimate the number $1/4.002$.

7. Use a linear approximation (or differentials) to estimate the number $\sqrt{99.8}$.

8. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
   (a) Use differentials to estimate the maximum error in the calculated area of the disk.
   (b) What is the relative error? What is the percentage error?

9. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.