Math 2A Single Variable Calculus Homework Questions 4

# 4 Applications of Derivatives

## 4.1 Maximum and Minimum Values

1. Find the critical points of $f(x) = x^3 + 6x^2 - 15x$.
2. Find the critical points of $h(p) = \dfrac{p-1}{p^2+4}$.
3. Find the critical points of $g(\theta) = 4\theta - \tan\theta$.
4. Find the absolute maximum and minimum values of $f(x) = 5 + 54x - 2x^3$ on the interval $[0, 4]$.
5. Find the absolute maximum and minimum values of $f(x) = x^3 - 6x^2 + 5$ on the interval $[-3, 5]$.
6. Find the absolute maximum and minimum values of $f(x) = \dfrac{x}{x^2 - x + 1}$ on the interval $[0, 3]$.
7. Find the absolute maximum and minimum values of $f(t) = \sqrt[3]{t}(8 - t)$ on the interval $[0, 8]$.
8. An object with weight $W$ is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is

   $$F = \frac{\mu W}{\mu \sin\theta + \cos\theta},$$

   where $\mu$ is a positive constant called the *coefficient of friction* and $0 \le \theta \le \pi/2$. Show that $F$ is minimized when $\tan\theta = \mu$.
9. Show that 5 is a critical number of the function

   $$g(x) = 2 + (x - 5)^3,$$

   but that $g$ does not have a local extreme value at $x = 5$.

## 4.2 The Mean Value Theorem

1. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but that there is no number $x$ in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?
2. Verify that the function $f(x) = x^3 - 3x + 2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 2]$, and find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.
3. Verify that the function $f(x) = 1/x$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 3]$, and find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.
4. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of $c$ such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

5. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

6. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

7. Suppose that $3 \leq f'(x) \leq 5$ for all values of $x$. Show that $18 \leq f(8) - f(2) \leq 30$.

8. Suppose that $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose also that $f(a) = g(a)$ and that $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. (*Hint: Apply the Mean Value Theorem to the function $h = f - g$.*)

9. At 2 pm a car's speedometer reads 30 mi/h. Ten minutes later it reads 50 mi/h. Prove that at some time between 2 and 2.10 pm the acceleration of the car is exactly 120 mi/h$^2$.

10. A number $a$ is called a *fixed point* of a function $f$ if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers $x$ then $f$ has at most one fixed point.

## 4.3 How Derivatives Affect the Shape of a Graph

1. Let $f(x) = 4x^3 + 3x^2 - 6x + 1$.
   (a) Find the intervals on which $f$ is increasing or decreasing.
   (b) Find the local maximum and minimum values of $f$.
   (c) Find the intervals of concavity and the inflection points.

2. Let $f(x) = \cos^2 x - 2\sin x$, $0 \leq x \leq 2\pi$.
   (a) Find the intervals on which $f$ is increasing or decreasing.
   (b) Find the local maximum and minimum values of $f$.
   (c) Find the intervals of concavity and the inflection points.

3. (a) Find the critical numbers of $f(x) = x^4(x-1)^3$.
   (b) What does the Second Derivative Test tell you about the behavior of $f$ at these critical numbers?
   (c) What does the First Derivative Test tell you?

4. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all $x$.
   (a) Sketch a possible graph for $f$.
   (b) How many solutions does the equation $f(x) = 0$ have? Why?
   (c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

5. Let $f(x) = 36x + 3x^2 - 2x^3$.
   (a) Find the intervals of increase or decrease.
   (b) Find the local maximum and minimum values.
   (c) Find the intervals of concavity and the inflection points.
   (d) Use the information from parts (a-c) to sketch the graph.

6. Let $G(x) = 5x^{2/3} - 2x^{5/3}$.

   (a) Find the intervals of increase or decrease.
   (b) Find the local maximum and minimum values.
   (c) Find the intervals of concavity and the inflection points.
   (d) Use the information from parts (a-c) to sketch the graph.

7. Let $S(x) = x - \sin x, 0 \leq x \leq 4\pi$.

   (a) Find the intervals of increase or decrease.
   (b) Find the local maximum and minimum values.
   (c) Find the intervals of concavity and the inflection points.
   (d) Use the information from parts (a-c) to sketch the graph.

8. Use the methods of this section to sketch the graph of $f(x) = x^3 - 3a^2x + 2a^3$, where $a$ is a positive constant. What do the members of this family have in common? How do they differ from each other?

9. Show that the curve $y = (1+x)/(1+x^2)$ has three points of inflection and that they all lie on one straight line.

10. (HARD!) The three cases of the First Derivative Test cover the most common, but not all, situations. Consider the following functions $f$, $g$ and $h$:

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} x^4 \left(2 + \sin \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

$$h(x) = \begin{cases} x^4 \left(-2 + \sin \frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Show that 0 is a critical number of all three functions but their derivatives change infinitely often on both sides of 0.
   (b) Show that $f$ has a neither a local maximum nor a local minimum at 0, $g$ has a local minimum, and $h$ has a local maximum.

### 4.4 Indeterminate Forms and L'Hôpital's Rule

1–11 Find the limit. Use l'Hôpital's rule where approropriate. If you can find the limit in an elementary way, do so. If l'Hôpital's rule doesn't app;ly, say why.

1. $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}$

2. $\displaystyle\lim_{x\to \infty} \frac{x^2 + x}{1 - 2x^2}$

3. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$

4. $\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$, where $m, n$ are any constants.

5. $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

6. $\lim_{x \to \infty} x \tan(\frac{1}{x})$

7. $\lim_{x \to 0} \csc x - \cot x$

8. $\lim_{x \to 0^+} (\tan 2x)^x$

9. $\lim_{x \to \infty} (e^x + x)^{1/x}$

10. $\lim_{x \to 1} (2 - x)^{\tan(\pi x/2)}$

11. $\lim_{x \to \infty} \left(\frac{2x - 3}{2x + 5}\right)^{2x+1}$

12. A metal cable has radius $r$ and is covered by electrical insulation, so that the distance from the center of the cable to the exterior of the insulation is $R$. The velocity of an electrical impulse in the cable is

$$v = -c \left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where $c$ is a positive constant. Find the following limits and interpret your answers.

    (a) $\lim_{R \to r^+} v$

    (b) $\lim_{r \to 0^+} v$

### 4.5 Summary of Curve Sketching

1. Sketch the curve $y = x^4 - 8x^2 + 8$.

2. Sketch the curve $y = \frac{x^2 - 4}{x^2 - 2x}$.

3. Sketch the curve $y = 1 + \frac{1}{x} + \frac{1}{x^2}$.

4. Sketch the curve $y = x\sqrt{2 - x^2}$.

5. Sketch the curve $y = x + \cos x$.

6. Sketch the curve $y = \frac{\sin x}{2 + \cos x}$.

7. In the theory of relativity, the energy of a particle is

$$E = \sqrt{m_0^2 c^4 + h^2 c^2 / \lambda^2},$$

where $m_0$ is the rest mass of the particle, $\lambda$ is its wavelength, and $h$ is Planck's constant. Sketch the graph of $E$ as a function of $\lambda$. What does the graph say about the energy?

8. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the dquare of the distance between them. At positions 0 and 2 are placed particles with charge +1, while, between them, at $x$ is a particle of charge -1. The net force acting on the middle particle is thus

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2}, \quad 0 < x < 2,$$

where $k$ is a positive constant. Sketch a graph of the net force function. What does the graph say about the force?

9. Sketch the curve $y = \dfrac{1+5x-2x^2}{x-2}$, taking care to find the equation of the slant asymptote.

10. Sketch the curve $y = \dfrac{(x+1)^3}{(x-1)^2}$, taking care to find the equation of the slant asymptote.

11. Let $f(x) = (x^3+1)/x$. Show that

$$\lim_{x \to \pm\infty} (f(x) - x^2) = 0.$$

This shows that the graph of $f$ approaches the graph of $y = x^2$ for large $x$ and we say that $f$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of $f$.

12. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

## 4.7 Optimization Problems

1. Find two numbers whose difference is 100 and whose product is a minimum.

2. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

3. Find the dimensions of a rectangle with area 1000 $m^2$ whose perimeter is as small as possible.

4. A box with a square base and open top must have a volume of 32,000 $cm^3$. Find the dimensions of the box that minimize the amount of material used.

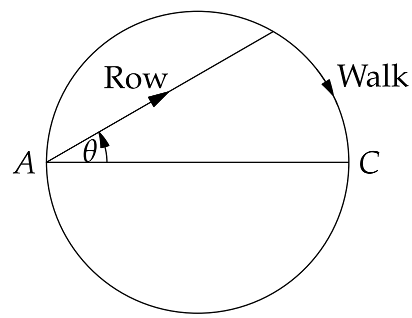
5. (a) Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.

   (b) Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

6. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

7. A Norman window has the shape of a rectangle surmounted by a semicircle (thus the diameter of the semicircle is the width of the rectangle). If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

8. A woman at point $A$ on the shore of a circular lake with radius 2 mi wants to arrive at the point $C$ diametrically opposite $A$ on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at 2 mi/h. How should she proceed?




9. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit with remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

## 4.9 Anti-derivatives

4-16: Find the most general anti-derivative of the following function; check your answer by differentiation.

1. $f(x) = 8x^9 - 3x^6 + 12x^3$

2. $f(x) = x^{3.4} - 2x^{\sqrt{2}-1}$

3. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$

4. $f(\theta) = 6\theta^2 - 7\sec^2\theta$

5. Find the anti-derivative $F(x)$ of $f(x) = x + 2\sin x$ that satisfies the condition $F(0) = -6$.

6. Find $f$ if $f''(x) = x^6 - 4x^4 + x + 1$.

7. Find $f$ if $f'(x) = 5x^4 - 3x^2 + 4$ with $f(-1) = 2$.

8. Find $f$ if $f''(x) = 8x^3 + 5$, $f(1) = 0$ and $f'(1) = 8$.

9. Find a function $f$ such tht $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of $f$.

10. A particle is moving with data $v(t) = 1.5\sqrt{t}$ and $s(4) = 10$. Find the position of the particle at time $t$.

11. A particle is moving with data $a(t) = 3\cos t - 2\sin t$, $s(0) = 0$ and $v(0) = 4$. Find the position of the particle at time $t$.

12. A particle is moving with data $a(t) = t^2 - 4t + 6$, $s(0) = 0$ and $s(1) = 20$. Find the position of the particle at time $t$.

13. A car is traveling at 50 mi/h when the brakes are fully applied, producing a deceleration of 22 $\text{ft/s}^2$. What is the distance traveled before the car comes to a stop?