1.3 New Functions from Old Functions

Let f be a function whose graph is known, and let c be a positive constant. The following basic transformations of graphs should be familiar.

- $y = f(x) \pm c$: shifts graph up/down
- $y = f(x \mp c)$: shift graph left/right
- $y = cf(x)$: stretch vertically by a factor c
- $y = f(cx)$: compress horizontally by a factor c
- $y = -f(x)$: reflect in the x-axis
- $y = f(-x)$: reflect in the y-axis

For example $y = 8(x-3)^2 + 4$ will have the same parabolic shape as $y = x^2$ but will be stretched vertically, shifted 3 to the right, and 4 up.

Composition of functions $f \circ g$: do g first, then f. For example, let $f(x) = \sqrt{x-4}$ and $g(x) = x^2 + 3x$, then

$$f \circ g(x) = f\big(g(x)\big) = f(x^2+3x) = \sqrt{x^2+3x-4}$$

and

$$g \circ f(x) = g\big(f(x)\big) = g(\sqrt{x-4}) = (\sqrt{x-4})^2 + 3\sqrt{x-4} = x - 4 + 3\sqrt{x-4}$$

You should be able to check that the domains and ranges of these functions are

Function	Domain	Range
f	$[4,\infty)$	$[0,\infty)$
g	$\mathbb{R}$	$[-\frac{9}{4},\infty)$
$f \circ g$	$(-\infty,-4] \cup [1,\infty)$	$[0,\infty)$
$g \circ f$	$[4,\infty)$	$[0,\infty)$

The domain of $f \circ g$ is probably the most difficult to see. Completing the square, we see that

$$f \circ g(x) = \sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{25}{4}}$$

Therefore

$$\begin{aligned} x \in \text{dom}(f \circ g) &\iff \left(x+\frac{3}{2}\right)^2 \geq \frac{25}{4} \iff \left|x+\frac{3}{2}\right| \geq \frac{5}{2} \\ &\iff x+\frac{3}{2} \geq \frac{5}{2} \quad \text{or} \quad x+\frac{3}{2} \leq -\frac{5}{2} \\ &\iff x \geq 1 \quad \text{or} \quad x \leq -4 \end{aligned}$$

Homework

1. Let $f(x) = \sec x$ and $g(x) = (x^2-1)^{-1/2}$. Find $f \circ g$, $g \circ f$ and their domains. Simplify if you can.
2. Repeat the question for $f(x) = \sin x$. What is the problem?