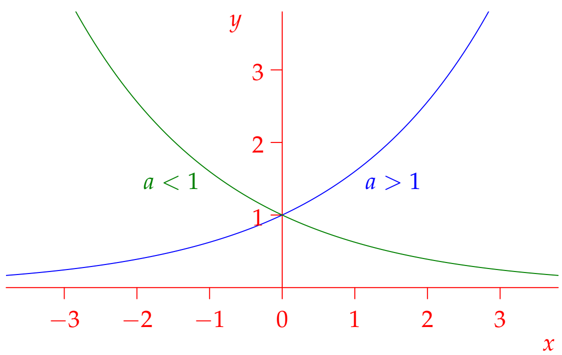
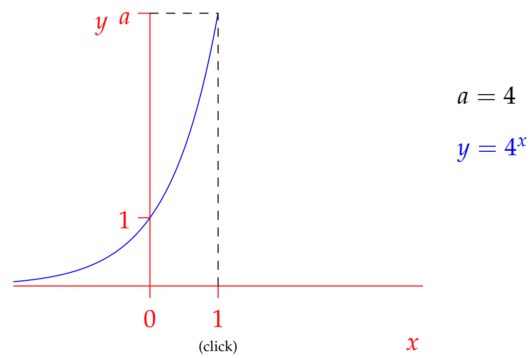
1.4 Exponential Functions

Definition. *An* exponential function *is a function of the form $f(x) = a^x$ where $a \neq 1$ is a positive constant. We call a the* base.

The graphs when $a < 1$ and $a > 1$ are different.





Should be able to move graphs around using rules from earlier. For example, sketch the graph of $f(x) = 2\left((\frac{1}{3})^x - 1\right)\ldots$

Exponent Laws

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

Negative Exponents Since $a^{-1} = \dfrac{1}{a}$, it is common to write decreasing exponential functions with a negative exponent. E.g.

$$f(x) = 3^{-x} = \left(\frac{1}{3}\right)^x$$

Growth Rates Exponential functions are important when the rate of growth of a quantity is proportional to the quantity itself. For example, consider the function $f(x) = 2^x$ and $g(x) = 3^x$.

x	0		1		2		3		4		5		6		7
$f(x) = 2^x$	1		2		4		8		16		32		64		128
Difference		1		2		4		8		16		32		64	

Notice how the difference between successive terms is proportional to the value of the function. Repeating the exercise with $g(x) = 3^x$ we obtain

x	0		1		2		3		4		5		6		7
$g(x) = 3^x$	1		3		9		27		81		243		729		2187
Difference		2		6		18		54		162		486		1458	

Applications For example, if a is a constant and P_0 is some initial population, then

$$P(t) = P_0 a^t$$

would be a basic model for population growth.

Example A population of rabbits doubles every 6 months. If the initial population is 100 rabits, then the number of rabbits at after t years is

$$P(t) = 100 \cdot 4^t$$

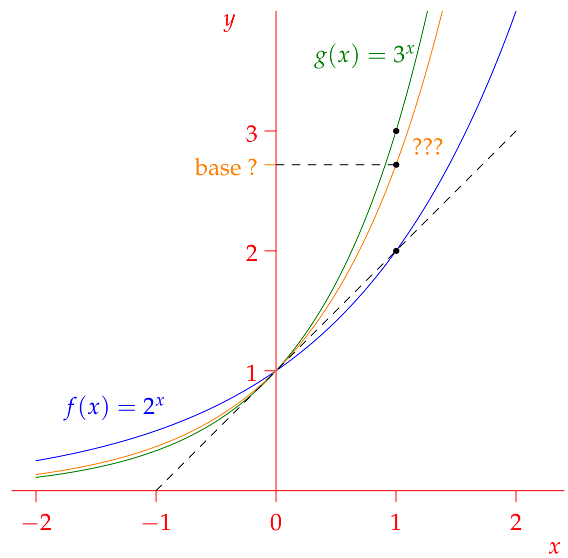
The Natural Exponential Function

Consider the graphs of $f(x) = 2^x$ and $g(x) = 3^x$ and their growth rates at $x = 0$.

The dashed line has gradient 1. 2^x clearly increases slower than this at $x = 0$, while 3^x increases more quickly.

Is there an exponential function, the orange curve, whose gradient at $x = 0$ is precisely 1? If so, then its base must be somewhere between 2 and 3.

The answer, of course, is yes...



Definition. *The* natural exponential function $\exp(x) = e^x$ *is the exponential function whose gradient is 1 at* $x = 0$. *Its base* e *is approximately* 2.71828.

e is an irrational number: like π, its decimal expansion goes on forever, without any pattern. Whenever you see it in a calculation, just leave it there, as you would π or $\sqrt{2}$. For example

$$e^4 \cdot (2e)^{-3} = \frac{1}{8}e$$

Because of its calculus benefits, scientists use the natural exponential almost exclusively. In the next section we will see how to convert any exponential function into an expression involving the natural exponential.

For example, using current data on world birth and death rates:

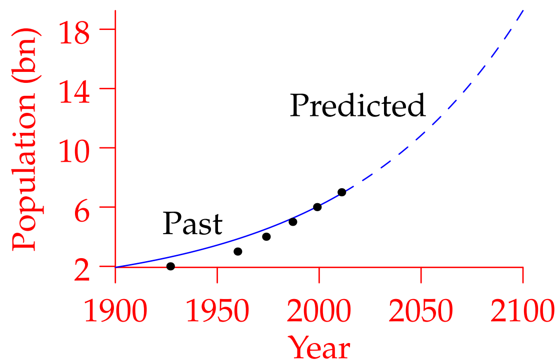
Birth Rate β = number of births per year, per person, currently[1] $\beta \approx 0.02$

Death Rate δ = number of deaths per year, per person, currently $\delta \approx 0.0085$

the model $P(t) = P_0 e^{(\beta-\delta)t} = P_0 e^{0.0115t}$ can be obtained. The model is so simple (just plug in β and δ) precisely because we are using the natural exponential. Any other base would make things messier.

[1] I.e. 20 out of every 1000 people give birth each year

Just to reassure you, there is evidence that the rate of population growth is reducing. Compare the graph, choosing P_0 so that the current world population is $\approx$ 7 billion, with historical data (the dots), and we see that the growth rate of the population was higher in the last century. We therefore shouldn't expect the population in 2100 to be as high as the graph suggests!



e and compound interest

The number e was discovered by Jacob Bernoulli during his investigations into compound interest. Here is how.

Suppose you place \$1 in a (very generous) bank account paying interest at 100% per year. At the end of the year you will have \$2. But what if the interest is paid more frequently?

- 50% paid twice per year: After half a year you are paid 50 cents, giving you \$1.50. This sum will now earn interest for six months, leaving you with $\$1.5^2 = \$2.25$ at the end of the year.
- 25% paid four times per year: the result is $\$1.25^4 \approx \$2.44$.
- $\frac{100}{12}\%$ paid monthly: at the end of the year you will have $\$\left(1+\frac{1}{12}\right)^{12} \approx \$2.61$.
- Daily: $\$(1+\frac{1}{365})^{365} = 2.714567\ldots$
- Every hour: $\$(1+\frac{1}{8760})^{8760} = 2.71812\ldots$

It seems that the more often you are paid, the closer you get to having $\$e$ at the end of the year. In modern times we consider this a definition of the number e: indeed we will later write

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Homework

1. In our model $P(t) = P_0e^{0.0115t}$, if the current world population is $P(2015) = 7\times 10^9$, what is P_0, and what would be the predicted population in 2100?
2. Let $f(x) = a^x$ for some constant $a > 1$ and let $h > 0$ be a small number.
   (a) Show that the difference $f(x+h) - f(x)$ is always proportional to $f(x)$, regardless of h.
   (b) Draw a picture which shows that the gradient of the curve $y = f(x)$ at $x = 0$ is approximated by the value $\dfrac{f(h)-f(0)}{h}$, when h is very small. Argue that the approximation improves, the smaller we make h.
   (c) Argue that if $h\approx 0$, then $\dfrac{e^h-1}{h}\approx 1$. Later this will be written $\lim\limits_{h\to 0}\dfrac{e^h-1}{h} = 1$.