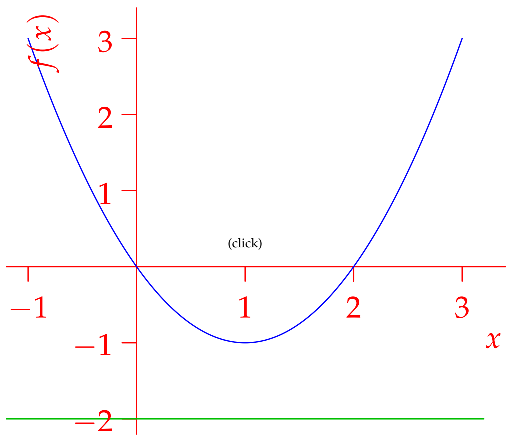
1.5 Inverse Functions and Logarithms

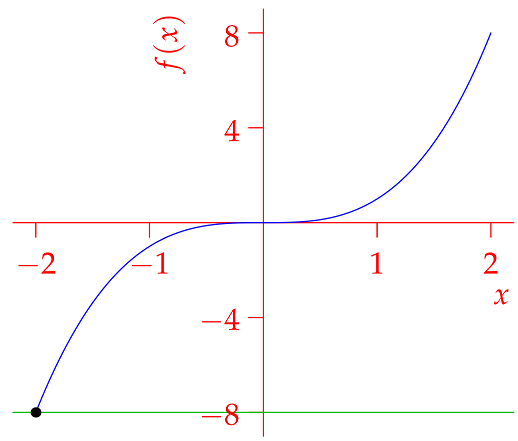
Definition. *f is 1–1 if never takes same value twice:* $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

This is equivalent to the *horizontal line test*: A function is 1–1 if no horizontal line intersects its graph more than once. For example:

1. $f(x) = x^2 - 2x$ is not 1–1 since $f(0) = f(2)$.
2. $f(x) = x^3$ is 1–1 since $x_1 \neq x_2 \implies x_1^3 \neq x_2^3$.



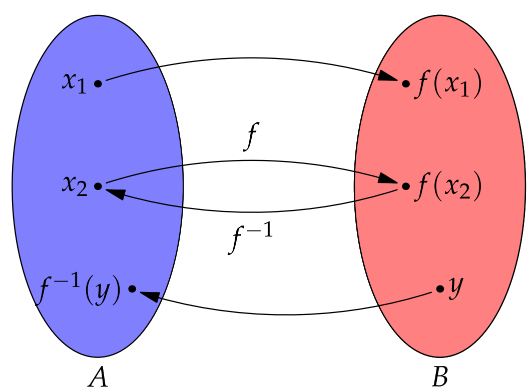
$f(x) = x^2 - 2x$ fails test



$f(x) = x^3$ passes test

Definition. *If f has domain A, range B, and is 1–1, then it has an* inverse function f^{-1} *with domain B and range A, defined by*

$$f^{-1}(y) = x \iff f(x) = y$$



Note that the domain and range swap over:

- Domain of f = Range of f^{-1}
- Range of f = Domain of f^{-1}

Cancellation laws

$$f(f^{-1}(y)) = y \qquad f^{-1}(f(x)) = x$$

WARNING! $^{-1}$ is not an exponent! Note the difference in the following:

- $f^{-1}(x)$ = the *value* of the inverse function f^{-1} applied to x.
- $(f(x))^{-1} = \frac{1}{f(x)}$ is not the same thing!

For example, if $f(x) = x^2$ with domain and range $[0, \infty)$, then $f^{-1}(x) = \sqrt{x}$. Notice that

$$(f(x))^{-1} = \frac{1}{x^2} \neq \sqrt{x} = f^{-1}(x)$$

How to find an inverse function

1. Check that f is 1–1.
2. Solve $y = f(x)$ for x in terms of y.
3. Interchange x and y.

It is only in step 3 that anything happens to x and y. Since we are swapping them, it is immediate that the graph of $y = f^{-1}(x)$ is simply that of $y = f(x)$ reflected in the line $y = x$.

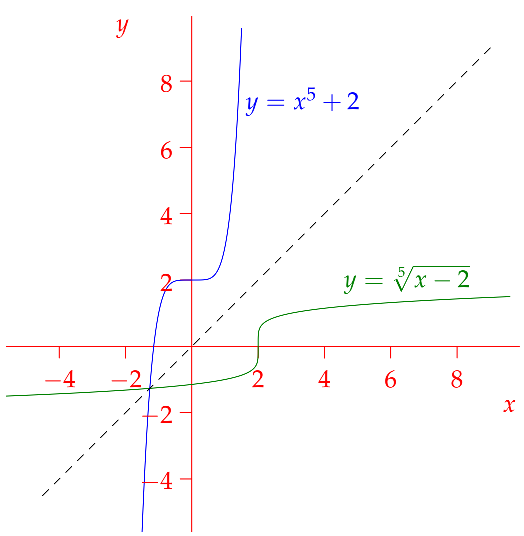
Examples Check the computations.

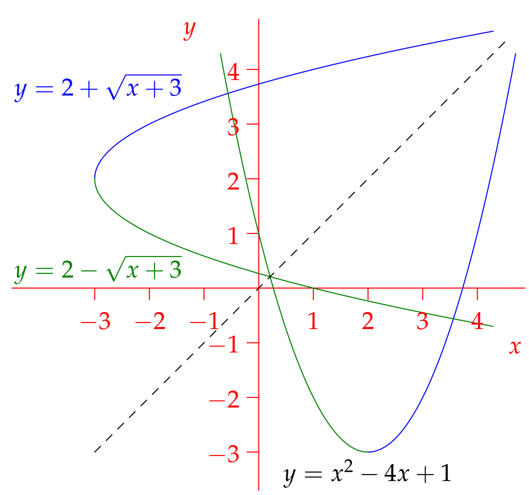
$f(x)$	$\operatorname{dom}(f)$	$\operatorname{range}(f)$	$f^{-1}(x)$
x^3	$\mathbb{R}$	$\mathbb{R}$	$x^{1/3} = \sqrt[3]{x}$
x^2	$[0,\infty)$	$[0,\infty)$	$\sqrt{x}$
x^2	$(-\infty,0]$	$[0,\infty)$	$-\sqrt{x}$
x^5+7	$\mathbb{R}$	$\mathbb{R}$	$\sqrt[5]{x-7}$
x^2-4x+1	$[2,\infty)$	$[-3,\infty)$	$2+\sqrt{x+3}$
x^2-4x+1	$(-\infty,2]$	$[-3,\infty)$	$2-\sqrt{x+3}$

For the last two examples, note that $y = x^2 - 4x + 1 = (x-2)^2 - 3$ is the standard parabola shifted right by 2 and down by 3. There are two maximal domains on which $y = f(x)$ is 1–1, namely $[2,\infty)$ (f increasing) and $(-\infty,2]$ (f decreasing). Suppose we take the second, with $\operatorname{dom}(f) = (-\infty,2]$. Following the procedure above:

1. f is 1–1 since it is decreasing.
2. $y = (x-2)^2 - 3 \implies y+3 = (x-2)^2 \implies x - 2 = -\sqrt{y+3}$. We take the *negative* square root since x, being in the *domain* of f, must be less than or equal to 2.
3. $y = 2 - \sqrt{x+3}$, as required.

The last three examples are drawn below: in the second picture, curves with the same color are mutual inverses.





Logarithmic Functions

If $a \neq 1$, then the exponential function $f(x) = a^x$ is 1–1. Indeed:

- f is *increasing* if $a > 1$
- f is *decreasing* if $a < 1$

Definition. *If* $0 < a < 1$ *or* $a > 1$*, define the* logarithm with base a *to be the inverse function to* $f(x) = a^x$*. We write*

$$f^{-1}(x) = \log_a x$$

The natural logarithm *is the logarithm with base* e*, the inverse of* $f(x) = e^x$*. We write*

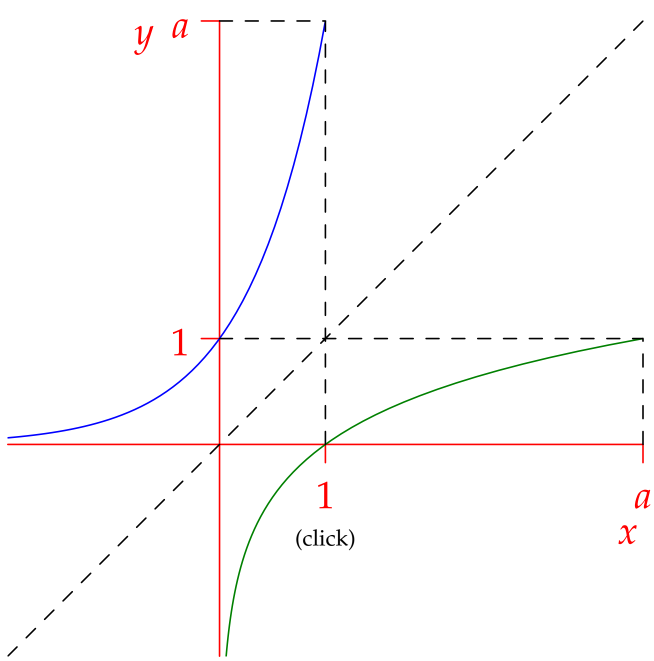
$$\ln x = \log_e x$$

Since all exponential functions have domain $\mathbb{R}$ and range $(0, \infty)$, it follows that all logarithms have domain $(0, \infty)$ and range $\mathbb{R}$.

Cancellation Laws Note that $y = a^x \iff x = \log_a y$. The cancellation laws for inverse functions read

$$\log_a(a^x) = x, \qquad a^{\log_a x} = x$$

Graphs $y = a^x$ and $y = \log_a x$ are drawn for different bases a



$a = 4$

$y = 4^x$

$y = \log_4 x$

Logarithm Laws These follow directly from the exponential laws.

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a(x^r) = r \log_a x$

Proof of law 1. We use the exponential and cancellation laws as above. Let $X = \log_a x$ and $Y = \log_a Y$, then:

$$\begin{aligned} \log_a x + \log_a y = X + Y &= \log_a(a^{X+Y}) && \text{(cancellation law)} \\ &= \log_a(a^X a^Y) && \text{(exponential law)} \\ &= \log_a(a^{\log_a x} a^{\log_a y}) \\ &= \log_a(xy) && \text{(cancellation law)} \end{aligned}$$

■

Solving Equations Logarithmic functions can easily be used to solve equation. For example:

1. $3^{5x+2} - 1 = 5 \implies 3^{5x+2} = 6 \implies 5x + 2 = \log_3 6 \implies = \frac{1}{5}(\log_3 6 - 2)$
2. $4^{x^2-3} = 8 \implies x = \pm\frac{3}{\sqrt{2}}$ (take $\log_2$ of both sides...)

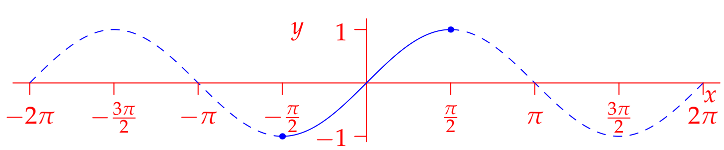
Change of base Too many logarithms: what to do?! Most calculators can cope with only the natural logarithm and the logarithm base 10. To evaluate, all other logarithms are typically converted to one of these. In calculus, any conversions will be to the natural logarithm.

Theorem. $\log_a x = \frac{\ln x}{\ln a}$

To approximate the solution to the first example, you would compute $x = \frac{1}{5}\left(\frac{\ln 6}{\ln 3} - 2\right) \approx -0.0738$.

Inverse Trigonometric Functions

The basic trigonometric functions have inverses, *provided you restrict the domain so that the original fucntion is 1–1!* The standard choices of domain are as follows.

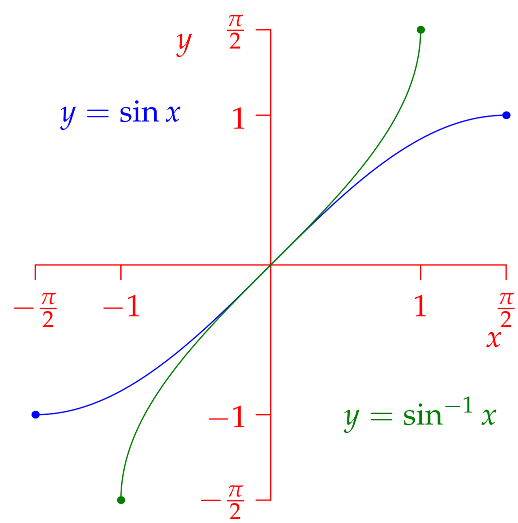


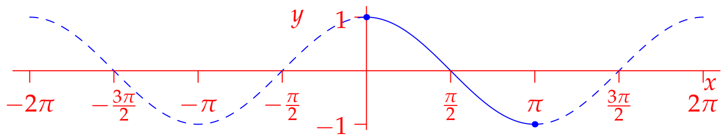
$f(x) = \sin x$ is 1–1 on domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Inverse function $f^{-1}(x) = \arcsin x = \sin^{-1} x$

Domain $\operatorname{dom}(\arcsin) = [-1, 1]$

Range $\operatorname{range}(\arcsin) = [-\frac{\pi}{2}, \frac{\pi}{2}]$



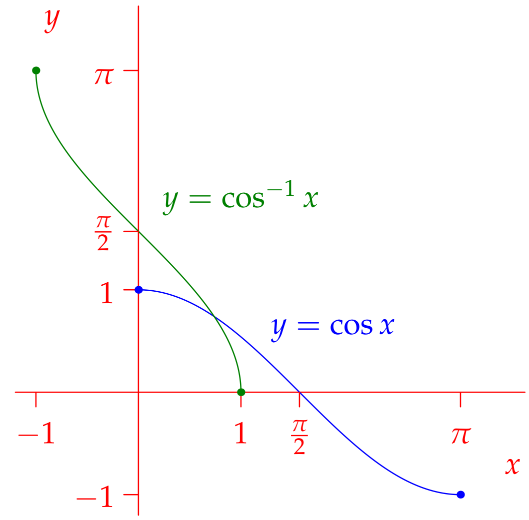


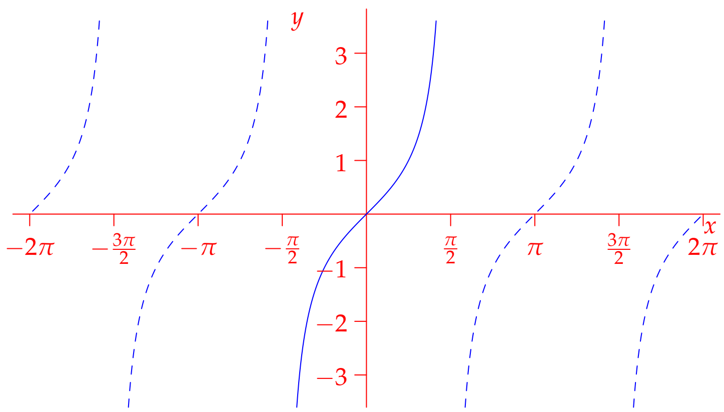
$f(x) = \cos x$ is 1–1 on domain $[0, \pi]$

Inverse function $f^{-1}(x) = \arccos x = \cos^{-1} x$

Domain $\text{dom}(\arccos) = [-1, 1]$

Range $\text{range}(\arccos) = [0, \pi]$



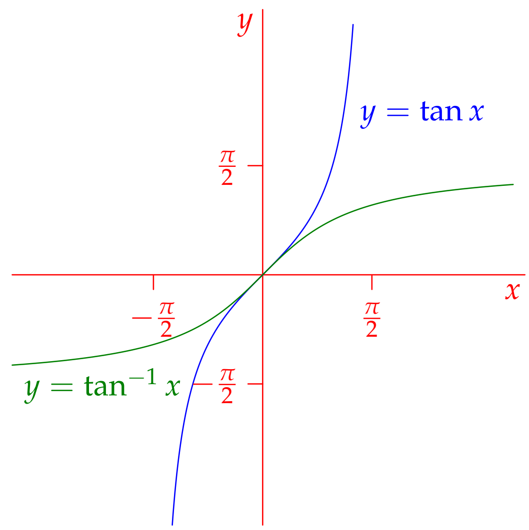


$f(x) = \tan x$ is 1–1 on domain $(-\frac{\pi}{2}, \frac{\pi}{2})$

Inverse function $f^{-1}(x) = \arctan x = \tan^{-1} x$

Domain $\text{dom}(\arctan) = \mathbb{R}$

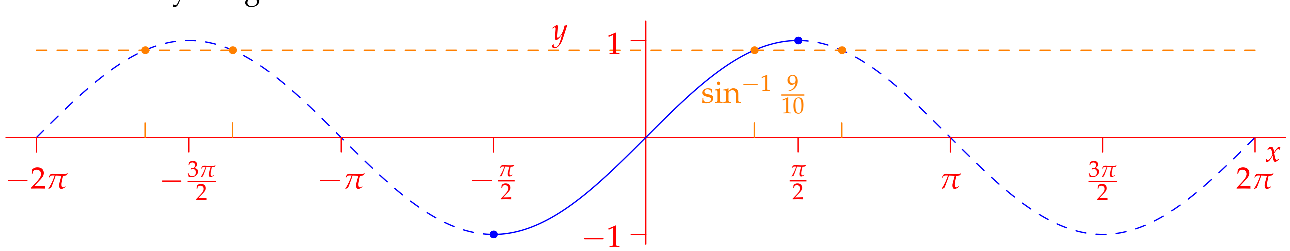
Range $\text{range}(\arcsin) = (-\frac{\pi}{2}, \frac{\pi}{2})$



Solving equations If you know the graphs, you can solve equations. For example, to solve the equation $\sin x = \frac{9}{10}$, you need all of the solutions

$$x = \sin^{-1} \frac{9}{10} + 2\pi n, \quad \pi - \sin^{-1} \frac{9}{10} + 2\pi n$$

where n is any integer.



WARNING! Don't let the $\sin^{-1}$ notation confuse you. A positive integer exponent on a trig function means square/cube/etc. the *value*, e.g.,

$$\sin^2 x = (\sin x)^2$$

The only other exponent you will see on a trig function is a -1, denoting the *inverse function.* In particular,

$$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x \quad \text{is not the same as} \quad \sin^{-1} x$$

Homework

1. A large piece of paper is 0.1 mm thick. Suppose you cut the paper and place the two pieces on top of each other. Now cut the stack and place the pieces in a single stack, now four pieces thick. Repeat the cut and stack exercise. How many times must you do this until your stack reaches to the moon $\approx 385,000$ km?

2. Prove the second and third logarithm laws, similarly to the way we proved the first.

3. Prove the change of base formula by taking natural logarithms of $x = a^{\log_a x}$.