# 2 Limits and Derivatives

## 2.1 The Tangent and Velocity Problems

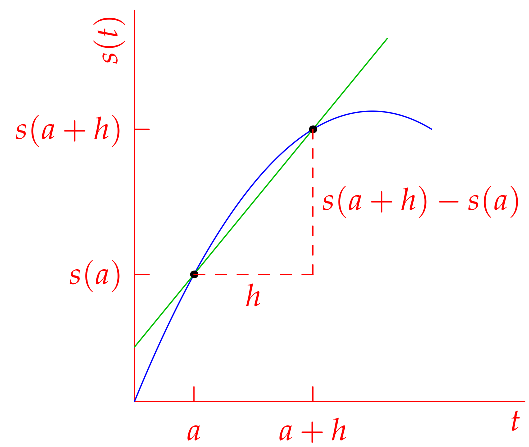
**Instantaneous Velocity** Differential Calculus was partly motivated by the idea of finding the instantaneous velocity of an object.

Let $s(t)$ measure the displacement (in meters) of a particle after $t$ seconds.

**Definition.** *The* average velocity *of the particle between times $t = a$ and $t = a + h$ seconds is*

$$v_{av} = \frac{\text{Distance}}{\text{Time}} = \frac{s(a+h) - s(a)}{h} \quad \text{m/s}$$

$v_{\text{av}}$ is the *gradient* of the green line.




If we compute the average velocity over smaller and smaller time intervals $h$, the average velocity should get closer to the *instantaneous velocity* of the particle at $t = a$

The green line is called a *secant* (cutting) line. As $h$ becomes smaller, the secant line will get closer to being *tangent* (touching) to the curve at $t = a$. The instantaneous velocity is therefore the gradient of the tangent line.

**Tangents** How do we find the tangent line to a curve at a given point?

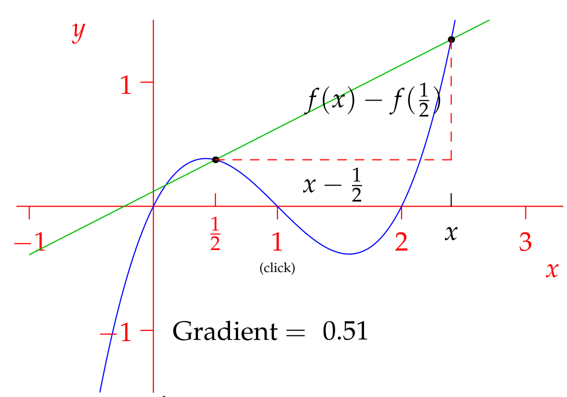
- To find the equation of a line, we need its *gradient*, and a point on the line. The point is given, so we just need the gradient.
- Approximate the tangent line by drawing secant lines and computing their gradients.
- The gradients of the secant lines will (hopefully!) approach some value as the secant lines are made to better approximate the tangent line.

**Example** Draw secant lines to find the gradient at $x = \frac{1}{2}$ of the graph of $f(x) = x^3 - 3x^2 + 2x$

The gradient of the secant line is

$$m_x = \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}$$

Measuring over smaller and smaller intervals $[\frac{1}{2}, x]$, the gradient seems to approach $-\frac{1}{4}$




If this approach is legitimate, then the tangent line has gradient $-\frac{1}{4}$ and passes through the point

$(\frac{1}{2}, f(\frac{1}{2})) = (\frac{1}{2}, \frac{3}{8})$. Its equation is therefore

$$y - \frac{3}{8} = -\frac{1}{4}\left(x - \frac{1}{2}\right) \iff y = -\frac{1}{4}x + \frac{1}{2}$$

To get any further, we need to formalize the limiting process.

**Homework**

1. Suppose that, after $t$ seconds, a particle has displacement $s(t) = 7t - 3t^2$ meters. Show that the average velocity over the time interval $[2, 2+h]$ is

   $$v_{\text{av}} = -5 - 3h \text{ m/s}$$

   Therefore compute the velocity of the particle at time $t = 2$.

2. Find the equation of the tangent line to $y = 7x - 3x^2$ at $x = 1$.