2.2 The Limit of a Function

The vague notion of approximating gradients and velocities has a precise mathematical description and terminology.

Definition. *Let a be a constant real number. Suppose that f is a function defined* nearby *a, but not necessarily* at *a. That is, for some small quantity h, the domain of f includes the set*

$$(a-h,a)\cup(a,a+h)$$

Now suppose that we can force $f(x)$ to be arbitrarily close to some value L simpky by requiring x to be sufficiently close to a. In such a case we say that L is the limit *of f as x approaches a and write*

$$\lim_{x\to a} f(x) = L$$

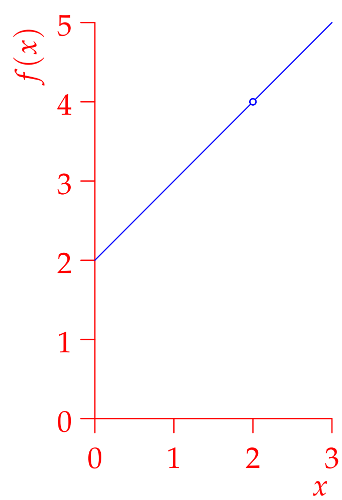
The strict definition of limit makes the concept of 'arbitrarily close' explicit. This level of detail is beyond the scope of this course.

Definition. *Let a be constant and suppose that f is defined near $x=a$. We say that $\lim_{x\to a} f(x) = L$ if: For any given $\varepsilon > 0$, there is some distance $\delta > 0$ for which any x closer to a than δ satisfies $|f(x)-L| < \varepsilon$.*

This discussion will be returned to in an upper-division Analysis course.

Indeterminate Forms The definition is most useful when an attempt to calculate $f(a)$ results in an *indeterminate form* such as $\frac{0}{0}$.

Example Find $\lim_{x\to 2} f(x)$ when $f(x) = \dfrac{x^2-4}{x-2}$



Evaluating directly yields the meaningless expression $\frac{0}{0}$. Indeed the function is not defined when $x=0$, its graph has a hole.
Taking values of x close to 2 we obtain

x	1	1.5	1.8	1.9	1.99	1.999
$f(x)$	3	3.5	3.8	3.9	3.99	3.999
x	3	2.5	2.2	2.1	2.01	2.001
$f(x)$	5	4.5	4.2	4.1	4.01	4.001

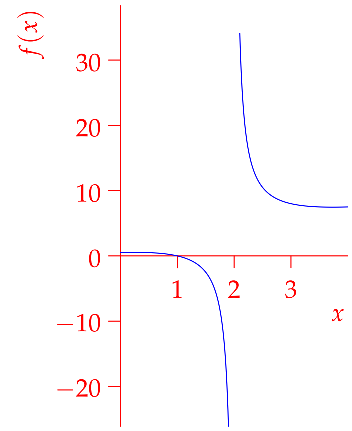
It seems reasonable to claim that $\lim_{x\to 2} f(x) = 4$.

In fact we can *prove* this assertion because the numerator is easy to factorize:

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x+2$$

whenever $x \neq 2$. It follows that as x approaches 2, $f(x)$ approaches 4.

A function with no limit Consider $f(x) = \dfrac{x^2-1}{x-2}$ as x approaches 2.



If we evaluate f at $x = 2$ we obtain the meaningless expression $\frac{3}{0}$. Taking values of x close to 2 we instead find

x	1	1.5	1.8	1.9	1.99
$f(x)$	0	-2.5	-11.2	-26.1	-296.01
x	3	2.5	2.2	2.1	2.01
$f(x)$	8	10.5	8.55	34.1	304.01

It appears that the values of $f(x)$:

$$\left.\begin{array}{c}\text{increase}\\ \text{decrease}\end{array}\right\} \text{ unboundedly as } x \text{ approaches 2 from } \left\{\begin{array}{l}\text{above}\\ \text{below}\end{array}\right.$$

Since the values of $f(x)$ are not getting closer to anything, we say that the limit of f at $x = 2$ *does not exist*. It is common to write

$$\lim_{x\to 2} f(x) = \text{DNE}$$

One-sided Limits

If we only consider values of x which are *greater* than a then we obtain the concept of *limit from above*.

Definition. *Suppose that the values $f(x)$ get arbitrarily close to L whenever x approaches a and $x > a$: we say that L is the* limit of f as x approaches a from above *and write*

$$\lim_{x\to a^+} f(x) = L$$

We similarly have the limit from below*:* $\lim_{x\to a^-} f(x) = L$

These are also known as *right-* and *left-sided* limits.

Theorem. $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L$ *and* $\lim_{x\to a^-} f(x) = L$.

Infinite Limits and Asymptotes

Definition. *Suppose that the values $f(x)$ get arbitrarily large whenever x approaches a: we write*

$$\lim_{x\to a} f(x) = \infty$$

The negative limit $\lim_{x\to a} f(x) = -\infty$ and the one-sided limits $\lim_{x\to a^\pm} f(x) = \pm\infty$ are similar.
If any of these limits are $\pm\infty$, we say that f has a vertical asymptote *at $x = a$.*

The function $f(x) = \frac{x^2-1}{x-2}$ discussed above has a vertical asymptote, since

$$\lim_{x\to 2^+} f(x) = \infty \quad \text{and} \quad \lim_{x\to 2^-} f(x) = -\infty$$

Examples

1. $f(x) = \dfrac{x-1}{2x^2+3x+1}$ has vertical asymptotes at $x = -1$ and $x = -\frac{1}{2}$ (factorize the denominator!)

2. $f(x) = \ln x$ has a vertical asymptote at $x = 0$. Indeed $\lim\limits_{x\to 0^+} \ln x = -\infty$.

3. $f(x) = \tan x$ has a vertical asymptote at every half-multiple of $\frac{\pi}{2}$.

Piecewise Functions Piecewise functions often have different left and right limits.

Example Let $f(x) = \begin{cases} 1+x & x < 1 \\ 4-x^2 & x \geq 1 \end{cases}$

It is easy to convince yourself that

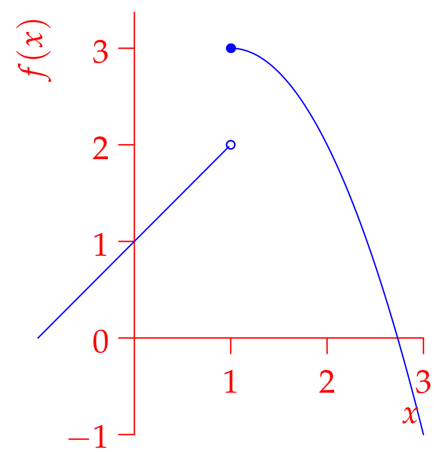
$$\lim_{x\to 1^-} f(x) = 2, \quad \text{and} \quad \lim_{x\to 1^+} f(x) = 3.$$

Consequently $\lim\limits_{x\to 1} f(x)$ does not exist.

The value $f(1) = 3$ is totally irrelevant, indeed the functions

$$g(x) = \begin{cases} 1+x & x \leq 1 \\ 4-x^2 & x > 1 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 1+x & x < 1 \\ 14 & x = 1 \\ 4-x^2 & x > 1 \end{cases}$$

have identical left and right limits to those of f.

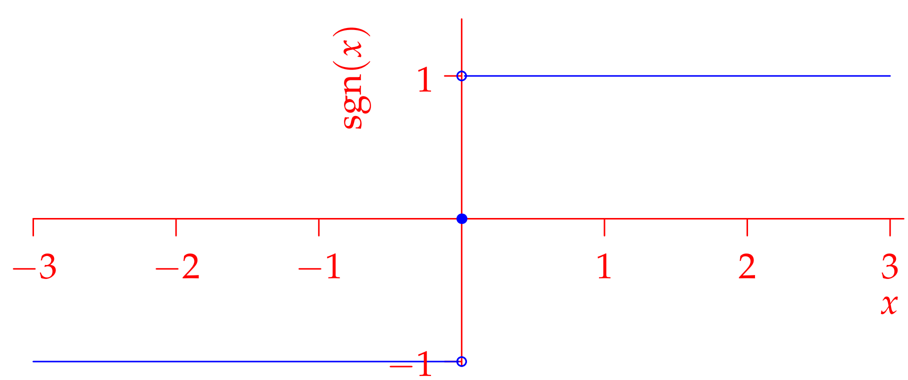


Three Famous Examples

1. The *Sign function* is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

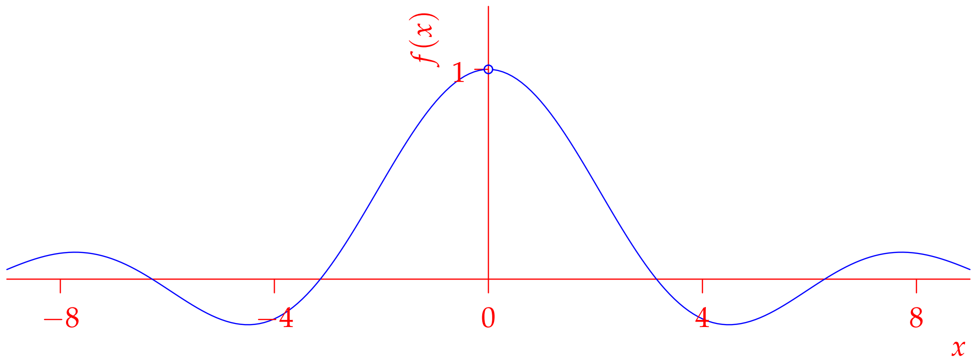
Clearly $\lim\limits_{x\to 0^-} \operatorname{sgn}(x) = -1 \neq 1 = \lim\limits_{x\to 0^+} \operatorname{sgn}(x)$ and so $\lim\limits_{x\to 0} \operatorname{sgn}(x)$ does not exist.



2. Consider $f(x) = \dfrac{\sin x}{x}$ as x approaches 0.

   Constructing a table of values suggests $\lim_{x\to 0} f(x) = 1$.

x	-1	-0.1	-0.01	0.01	0.1	1
$f(x)$	0.84147	0.99833	0.99998	0.99998	0.99833	0.84147



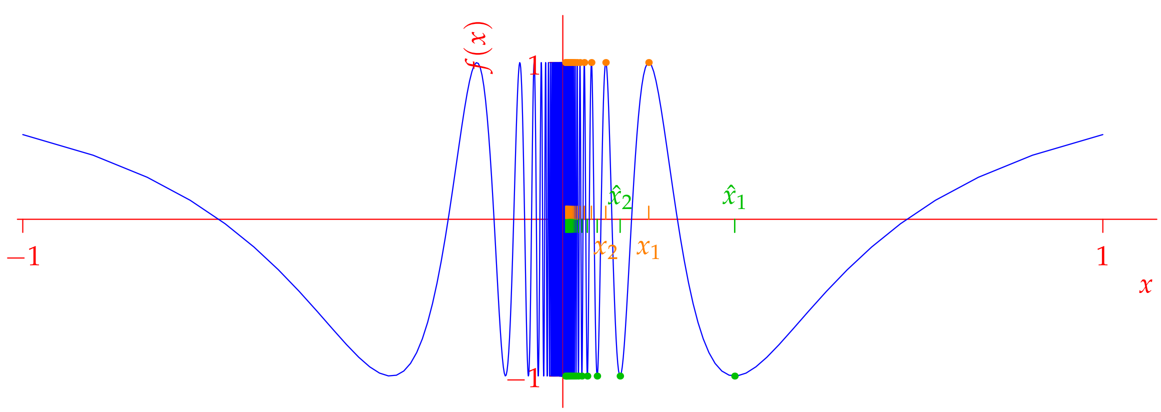
   This limit is *very* important in Calculus: we will return to it later.

3. (Difficult) The function $f(x) = \cos\left(\frac{1}{x}\right)$ is undefined at $x = 0$.

   The sequences $x_n = \dfrac{1}{2n\pi}$ and $\hat{x}_n = \dfrac{1}{(2n-1)\pi}$ where $n = 1, 2, 3, 4, 5, 6, \ldots$, both approach $x = 0$ from above, and yet

   $$f(x_n) = \cos(2n\pi) = 1 \quad \text{and} \quad f(\hat{x}_n) = \cos((2n-1)\pi) = -1$$

   The first sequence suggests $\lim_{x\to 0^+} f(x) = 1$, while the second suggests the limit is -1. *Neither* is true. The value of $f(x)$ oscillates infinitely many times between $y = \pm 1$ before reaching $x = 0$. The values of $f(x)$ do not get closer to anything, and so there is no limit at $x = 0$.



Homework

1. Find the vertical asymptotes of $f(x) = \dfrac{x-1}{2x^2 - x - 1}$. In particular, compute $\lim_{x\to 1} f(x)$ to show that f *does not* have a vertical asymptote at $x = 1$.